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SOME FIXED POINT RESULTS CONCERNING WITH USUAL METRIC SPACES FOR NON-SYMMETRIC RATIONAL EXPRESSIONS

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ABSTRACT

The present paper deals with some fixed point and common fixed point theorems in usual complete metric spaces for new symmetric rational expressions.

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Kew words: *Complete Metric Space, Usual Metric Space, Fixed Point, and Common Fixed Point.*

2. INTRODUCTION & PRELIMINARIES:

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non- linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear, therefore the fixed point methods specially Banach' contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [24] described about the application of Banach's contraction principle [2]. Browder [4] was the first mathematician to study non-expansive mappings. Meanwhile Brouwder [4] and Ghode [6] have independently proved a fixed point theorem for non-expansive mapping.

Many other mathematicians viz; Datson [5] Goebel [6], Goebel and Zlotkienwicz [8], Goebel, Kirk and Simi [9], Iseki [11], Singh and Chatterjee [22], Sharma and Rajput [21], Rajput and Naroliya [20] Pathak and Maity [18], Qureshi and Singh [19], Sharma and Bhagwan [23], Ahmad and Shakil [1], Shahzad and Udomene [24] have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk [15, 16 and 17] gave the comprehensive survey concerning fixed point theorems for non-expansive mappings. In the present paper we are proving some fixed point and common fixed point theorems for non contraction mappings in usual metric spaces for rational expressions which are motivated by Shrivastava, Dwivedi & Bhardwaj[25]

2.1 **Usual Metric Space:** A metric space (X,d) is said to be usual metric space if it is defined as 1

$d(x,y)=|x-y| \qquad \forall x,y\in X$ 1, 2.3 ……,

Theorem 3.1: - Let (X,d) be a complete usual metric space and $T:X\to X$ is self mapping;

such that $T^2=I$ ...(3.1.1)

$$d(Tx,Ty)\le \alpha\frac{d(x,Tx)d(y,Ty)d(x,Ty)+[d(x,y)]^3}{[d(x,y)]^2}$$
$$+\beta\frac{d(y,Ty)d(y,Tx)d(x,Ty)+[d(x,y)]^3}{[d(x,y)]^2}$$
$$+\gamma[d(x,Tx)+d(y,Ty)]+\delta[d(x,Ty)+d(y,Tx)]+\eta\ d(x,y) \qquad ...(3.1.2)$$

If $x \neq y,\ \forall x, y \in X$, with $10\alpha + 9\beta + 8\gamma + 5\delta + \eta < 4$ then T has unique fixed point in X.

Proof: Suppose x be any point in usual metric space X. Taking

$$y = \frac{1}{2}(T + I)x,\ \ z = T(y)$$

$$d(z,x) = d(Ty, T^2x) = |Ty - T^2x| \qquad \text{by def}^n$$

$$|Ty - T(Tx)| \le \alpha \frac{|y-Ty|\ |Tx-x|\ |y-x| + |y-Tx|^3}{|y-Tx|^2}$$

$$+ \beta \frac{|Tx-x|\ |Tx-Ty|\ |y-x| + |y-Tx|^3}{|y-Tx|^2}$$

$$+ \gamma\left[|y-Ty| + |Tx-x|\right] + \delta\left[|y-x| + |Tx-Ty|\right] + \eta\ |y-Tx|$$

$$\le \alpha \frac{|y-Ty|\ |Tx-x| \frac{1}{2}|x-Tx| + \frac{1}{8}|x-Tx|^3}{\frac{1}{4}|x-Tx|^2}$$

$$+ \beta \frac{|Tx-x|\ \left[|Tx-y| + |y-Ty|\right] \frac{1}{2}|x-Tx| + |x-Tx|^3}{\frac{1}{4}|x-Tx|^2}$$

$$+ \gamma\left[|y-Ty| + |Tx-x|\right] + \delta\left[\frac{1}{2}|x-Tx| + \frac{1}{2}|x-Tx| + |y-Ty|\right] + \eta \frac{1}{2}|x-Tx|$$

$$= |x-Tx|\left[\frac{\alpha}{2} + \beta + \gamma + \delta + \frac{\eta}{2}\right] + |y-Ty|\left[2\alpha + 2\beta + \gamma + \delta\right]$$

$$|z-x| \le \frac{1}{2}\left[\alpha + 2\beta + 2\gamma + 2\delta + \eta\right] |x-Tx| + |y-Ty|\left[2\alpha + 2\beta + \gamma + \delta\right] \qquad \ldots(3.1.3)$$

Now for $|u-x| = |2y - z - x| = |Tx - Ty|$

$$\le \alpha \frac{|x-Tx|\ |y-Ty|\ |x-Ty| + |x-y|^3}{|x-y|^2}$$

$$+ \beta \frac{|y-Ty|\ |y-Tx|\ |x-Ty| + |x-y|^3}{|x-y|^2}$$

$$+ \gamma\left[|x-Tx| + |y-Ty|\right] + \delta\left[|x-Ty| + |y-Tx|\right] + \eta\ |x-y|$$

$$|u-x| \le \alpha \frac{\frac{1}{2}|x-Tx|^2\ |y-Ty| + \frac{1}{8}|x-Tx|^3}{\frac{1}{4}|x-Tx|^2}$$

$$+ \beta \frac{|y-Ty|\ \frac{1}{2}|x-Tx|\ \frac{1}{2}|Tx-x| + |x-y|^3}{\frac{1}{4}|x-Tx|^2}$$

$$+ \gamma\left[|x-Tx| + |y-Ty|\right] + \delta\left[\frac{1}{2}|x-Tx| + \frac{1}{2}|x-Tx|\right] + \frac{\eta}{2}\ |x-Tx|$$

$$= \frac{1}{2}\left[\alpha + \beta + 2\gamma + \delta\right]|x-Tx| + (2\alpha + \beta + \gamma)|y-Ty| \qquad \ldots(3.1.4)$$

Now

$$|z-u| \le |z-x| + |x-u|$$

$$\le \frac{1}{2}|x-Tx|[\alpha+2\beta+2\gamma+2\delta+\eta] + |y-Ty|[2\alpha+2\beta+\gamma+\delta] + \frac{1}{2}|x-Tx|[\alpha+\beta+2\gamma+\delta]$$

$$+|y-Ty|[2\alpha+\beta+\gamma]$$

$$= \frac{1}{2}|x-Tx|[2\alpha+3\beta+4\gamma+3\delta+\eta] + |y-Ty|[4\alpha+3\beta+2\gamma+\delta] \qquad ...(3.1.5)$$

On the other hand

$$|z-u| = |Ty-(2y-z)|$$

$$= |Ty-2y+T(y)|$$

$$|z-u| = 2|Ty-y| \qquad ...(3.1.6)$$

Comparing $(3.1.5)$ and $(3.1.6)$

$$|Ty-y| \le s|Tx-x|$$

where $s = \dfrac{2\alpha+3\beta+4\gamma+3\delta+\eta}{4-(8\alpha+6\beta+4\gamma+2\delta)} < 1$

because $10\alpha+9\beta+8\gamma+5\delta+\eta < 4$

Let $R = \frac{1}{2}|T+I|$, then

$$d\left(R^2(x), R(x)\right) = \left|R^2(x) - R(x)\right|$$

$$= \left|R(R(x)) - R(x)\right|$$

$$= |R(y) - y|$$

$$= \frac{1}{2}|y-Ty|$$

$$< \frac{s}{2}|x-Tx|$$

i.e.

$$d\left(R^2(x), R(x)\right) < \frac{s}{2} d(x, Tx)$$

By the definition of *R*, we claim that $\{R_n(x)\}$ in a Cauchy sequence in *X*. By the completeness of $X, \{R^n(x)\}$ Converges to some element x_0 in *X*. So

$$\lim_{n\to\infty}\{R^n(x)\} = x_0$$

So $\{R(x_0)\} = x_0$

Hence $T(x_0) = x_0$

So x_0 is a fixed point of T.

Uniqueness: If possible let $y_0 \ne x_0$ is another fixed point of *T*.

$$d(x_0, y_0) = |x_0 - y_0|$$
$$\leq \alpha \frac{|x_0 - Tx_0|\,|y_0 - Ty_0|\,|x_0 - Ty_0| + |x_0 - y_0|^3}{|x_0 - y_0|^2}$$
$$+ \beta \frac{|y_0 - Ty_0|\,|y_0 - Tx_0|\,|x_0 - Ty_0| + |x_0 - y_0|^3}{|x_0 - y_0|^2}$$
$$+ \gamma\left[|x_0 - Tx_0| + |y_0 - Ty_0|\right] + \delta\left[|x_0 - Ty_0| + |y_0 - Tx_0|\right] + \eta\,|x_0 - y_0|$$
$$|x_0 - y_0| \leq (\alpha + \beta + 2\delta + \eta)|x_0 - y_0|$$
$$i.e.\ d(x_0, y_0) \leq (\alpha + \beta + 2\delta + \eta)\, d(x_0, y_0)$$

Which is a contradiction.

So $x_0 = y_0$

Hence fixed point is unique.

Now we prove common fixed point theorems for two mappings.

Theorem: 3.2 Let *K* be closed and convex subset of a complete Usual Metric Space *X*.
$Let\, T : K \to K$ and $G : K \to K$ satisfies the following conditions. *T* and *G* commute … (3.2.1)

$T^2 = I$ and $G^2 = I$ ...(3.2.2)

Where *I* denotes the identity mapping.

$$d(Tx,Ty) \leq \alpha \frac{d(Gx,Tx)\; d(Gy,Ty)\; d(Gx,Ty) + \left[d(Gx,Gy)\right]^3}{\left[d(Gx,Gy)\right]^2}$$
$$+ \beta \frac{d(Gy,Ty)\; d(Gy,Ty)\; d(Gx,Ty) + \left[d(Gx,Gy)\right]^3}{\left[d(Gx,Gy)\right]^2}$$
$$+ \gamma\left[d(Gx,Tx) + d(Gy,Ty)\right] + \delta\left[d(Gx,Ty) + d(Gy,Tx)\right] + \eta\; d(Gx,Gy)$$

For every $x, y \in X$, $\alpha, \beta, \gamma, \delta, \eta \in [0,1[$ with $x \neq y$ and $d(Gx,Gy) \neq 0$ and $10\alpha + 9\beta + 8\gamma + 5\delta + \eta < 4$ then *T* and *G* have unique common fixed point.

Proof: Suppose *x* is point in usual metric space *X*. We have taken non-contraction mapping. So it is clear that $(TG)^2 = I$

Now

$$d(Tx,Ty) = d(TG(Gx), TG(Gy)) = |TG(Gx) - TG(Gy)|$$
$$\leq \alpha \frac{|G(G^2x) - T(G^2x)|\,|G(G^2y) - T(G^2y)|\,|G(G^2x) - T(G^2y)| + |G(G^2x) - G(G^2y)|^3}{|G(G^2x) - G(G^2y)|^2}$$
$$+ \beta \frac{|G(G^2y) - T(G^2y)|\,|G(G^2y) - T(G^2x)|\,|G(G^2x) - T(G^2y)| + |G(G^2x) - G(G^2y)|^3}{|G(G^2x) - G(G^2y)|^2}$$
$$+ \gamma\left[|G(G^2x) - T(G^2x)| + |G(G^2y) - T(G^2y)|\right] + \delta\left[|G(G^2x) - T(G^2y)| + |G(G^2y) - T(G^2x)|\right]$$
$$+ \eta\,|G(G^2x) - G(G^2y)|$$

$$= \alpha \frac{|G(x)-TG(Gx)|\,|G(y)-TG(Gy)|\,|G(x)-TG(Gy)|+|G(x)-G(y)|^3}{|G(x)-G(y)|^2}$$
$$+\beta \frac{|G(y)-TG(Gy)|\,|G(y)-TG(Gx)|\,|G(x)-TG(Gy)|+|G(x)-G(y)|^3}{|G(x)-G(y)|^2}$$
$$+\gamma\left[|G(x)-TG(Gx)|+|G(y)-TG(Gy)|\right]$$
$$+\delta\left[|G(x)-TG(Gy)|+|G(y)-TG(Gx)|\right]+\eta\,|G(x)-G(y)|$$

Taking $G(x)=p,\ G(y)=q,$ where $p\neq q$

$$|TG(Gx)-TG(Gy)|=|TG(p)-TG(q)|$$
$$= \alpha \frac{|p-TG(p)|\,|q-TG(q)|\,|p-TG(q)|+|p-q|^3}{|p-q|^2}$$
$$+\beta \frac{|q-TG(q)|\,|q-TG(p)|\,|p-TG(q)|+|p-q|^3}{|p-q|^2}$$
$$+\gamma\left[|p-TG(p)|+|q-TG(q)|\right]$$
$$+\delta\left[|p-TG(q)|+|q-TG(p)|\right]+\eta\,|p-q|$$

Taking $TG=R$ we get

$$|R(p)-R(q)|\le \alpha \frac{|p-R(p)|\,|q-R(q)|\,|p-R(q)|+|p-q|^3}{|p-q|^2}$$
$$+\beta \frac{|q-R(q)|\,|q-R(p)|\,|p-R(q)|+|p-q|^3}{|p-q|^2}$$
$$+\gamma\left[|p-R(p)|+|q-R(q)|\right]$$
$$+\delta\left[|p-R(q)|+|q-R(p)|\right]+\eta\,|p-q|$$

It is clear by theorem (3.1) that $R=TG$ has at least one fixed point say x_0 in K. That is

$$R(x_0)=TG(x_0)=x_0$$

and so $T(TG)(x_0)=Tx_0$

or $T^2(Gx_0)=T(x_0)$

$$G(x_0)=T(x_0)$$

Now

$$d(Tx_0,x_0)=|Tx_0-x_0|$$
$$=|Tx_0-T^2(x_0)|$$
$$\le \alpha \frac{|Gx_0-Tx_0|\,|Gx_0-T(Tx_0)|\,|Gx_0-Tx_0|+|Gx_0-G(Tx_0)|^3}{|Gx_0-G(Tx_0)|^2}$$
$$+\beta \frac{|G(Tx_0)-T(Tx_0)|\,|G(Tx_0)-Tx_0|\,|Gx_0-T(Tx_0)|+|Gx_0-G(Tx_0)|^3}{|Gx_0-G(Tx_0)|^2}$$
$$+\gamma\left[|Gx_0-Tx_0|+|G(Tx_0)-T(Tx_0)|\right]$$
$$+\delta\left[|Gx_0-T(Tx_0)|+|G(Tx_0)-Tx_0|\right]$$
$$+\eta\,|Gx_0-G(Tx_0)|$$

Clearly

$$|Tx_0 - x_0| \le [\alpha+\beta+2\delta+\eta]|Tx_0 - x_0|$$

which is a contradication. Because $\alpha+\beta+2\gamma+\eta<1$

So $T(x_0)=x_0$

That is x_0 is the fixed point of *T*.

But $T(x_0)=Gx_0$ & $G(x_0)=x_0$

Hence it is the common fixed point of *T* and *G*.

Uniqueness: If possible let $y_0 \neq x_0$ is another common fixed point of *T* and *G* then

$$d(x_0,y_0)=|x_0-y_0|=|T^2(x_0)-T^2(y_0)|=|T(Tx_0)-T(Ty_0)|$$

$$\le \alpha \frac{|G(Tx_0)-T(Tx_0)|\,|G(Ty_0)-T(Ty_0)|\,|G(Tx_0)-T(Ty_0)|+|G(Tx_0)-G(Ty_0)|^3}{|G(Tx_0)-G(Ty_0)|^2}$$

$$+\beta \frac{|G(Ty_0)-T(Ty_0)|\,|G(Ty_0)-T(Tx_0)|\,|G(Tx_0)-T(Ty_0)|+|G(Tx_0)-G(Ty_0)|^3}{|G(Tx_0)-G(Ty_0)|^2}$$

$$+\gamma\left[|G(Tx_0)-T(Tx_0)|+|G(Ty_0)-T(Ty_0)|\right]$$

$$+\delta\left[|G(Tx_0)-T(Ty_0)|+|G(Ty_0)-T(Tx_0)|\right]$$

$$+\eta\,|G(Tx_0)-G(Ty_0)|$$

i.e.

$|x_0-y_0| \le [\alpha+\beta+2\delta+\eta]|x_0-y_0|$ but $\alpha+\beta+2\delta+\eta<1$

So $x_0=y_0$

i.e.

$d(x_0,y_0)<d(x_0,y_0)$

Which is a contradiction.

So $x_0=y_0$ *i.e.* common fixed point is unique.

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