



COMMUTATIVITY OF ALTERNATIVE PERIODIC RINGS WITH $(xy)^{n+1} - (yx)^{n+1} \in Z(R)$

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ABSTRACT

Let R be an Alternative periodic ring. In this paper, we show that an n -torsion free alternative ring R with identity satisfying $x^n y^n = y^n x^n$ and $(xy)^{n+1} - y^{n+1} x^{n+1} \in Z(R)$ is commutative. We also prove that if R is an n -torsion free alternative periodic ring such that $(xy)^{n+1} - (yx)^{n+1} \in Z(R)$, n is a fixed positive integer and $N(R)$ is commutative, then R is commutative.

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INTRODUCTION

Abu-Khuzam, Bell and Yaqub [2] proved that an n -torsion free ring R with identity such that, for all x, y in R , $x^n y^n = y^n x^n$ and $(xy)^{n+1} - x^{n+1} y^{n+1} \in Z(R)$ must be commutative. They also have shown that a periodic n -torsion free ring (not necessarily with identity) for which $(xy)^n - (yx)^n \in Z(R)$ is commutative provided that the nilpotents of R form a commutative set. In this paper, we show that an n -torsion free alternative ring R with identity satisfying $x^n y^n = y^n x^n$ and $(xy)^{n+1} - y^{n+1} x^{n+1} \in Z(R)$ is commutative. We also prove that if R is an n -torsion free alternative periodic ring such that $(xy)^{n+1} - (yx)^{n+1} \in Z(R)$, n is a fixed positive integer and $N(R)$ is commutative, then R is commutative.

PRELIMINARIES

Throughout this paper, R denotes an alternative periodic ring, $N(R)$ denotes the set of all nilpotent elements of R , $Z(R)$ the center of R , $C(R)$ the commutator ideal of R .

In order to prove our main results, we state the following well-known results.

Lemma: 1[6] Suppose that R is a ring with identity 1. If $x^m [x, y] = 0$ and $(x + 1)^m [x, y] = 0$, for some x, y in R and some integer $m > 0$, then $[x, y] = 0$. A similar statement holds if we assume $[x, y] x^m = 0$ and $[x, y] (x + 1)^m = 0$ instead.

Lemma: 2[4] Let R be an n -torsion free ring with identity 1 such that $[x^n, y^n] = 0$ for all x, y in R . Let $N(R)$ denote the set of nilpotent elements of R . Then

- (i) $a \in N(R), x \in R$ imply $[a, x^n] = 0$,
- (ii) $a \in N(R), b \in N(R)$ imply $[a, b] = 0$.

Lemma: 3 Let R be a periodic ring such that $N(R)$ is commutative. Then the commutator ideal of R is nil and $N(R)$ forms an ideal of R .

Proof: Let $N(R)$ denote the set of nilpotent elements of R and we use the standard argument for the commutative case to show that $u_1 - u_2 \in N(R)$, whenever $u_1, u_2 \in N(R)$.

We prove this by induction on k . That is, if $u_k = 0$ and $r \in R$ then it is true that $(ur)^k = (ru)^k = 0$.

Consider the case $k = 2$.

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Let u be an element of R such that $u^2 = 0$ and let j be a positive integer for which $(ur)^j = e$ is idempotent (possibly zero). Then $re - ere$ is nilpotent and hence commutes with u .

$$\text{That is, } ur(ur)^j - u(ur)^j r(ur)^j = r(ur)^j u - (ur)^j r(ur)^j u. \quad (1)$$

and multiplying on the right of (1) by u gives $ur(ur)^j u = 0$.

So that $(ur)^{j+2} = (ru)^{j+2} = 0$. It follows that u commutes with both ur and ru .

So $(ur)^2 = (ru)^2 = 0$. Now suppose that the result holds for all y with $y^m = 0, m < k$.

Suppose $u^k = 0, k \geq 3$. Determining j as above and multiplying (1) by r on the left and u on the right, we get

$$(ru)^{j+2} = ru^2s + tu^2 \quad (2)$$

where s and t are elements of the subring generated by r and u . Since $(u^2)^{k-1} = 0$, the inductive hypothesis implies both ru^2s and tu^2 are nilpotent. Therefore (2) shows that ru and ur are nilpotent. Again invoking the fact that u must commute with ur and ru , we see that $(ur)^k = (ru)^k = 0$. Hence $N(R)$ is an ideal.

MAIN RESULTS

Theorem: 1 Let R be an alternative with identity and n be a fixed positive integer. Suppose that R is n -torsion free and that for all x, y in R ,

$$x^n y^n = y^n x^n, \quad (3)$$

$$\text{and } (xy)^{n+1} - y^{n+1} x^{n+1} \in Z(R). \quad (4)$$

Then R is commutative.

Proof: By (3), we have $[x^n, y^n] = 0$ for all x, y in R and by [5], the commutator ideal is nil. This implies that the set of nilpotent elements $N(R)$ forms an ideal.

Hence by Lemma 2(ii), $N(R)$ is a commutative ideal.

$$\text{This implies that } N^2(R) \subseteq Z(R). \quad (5)$$

$$\text{By (4), we have } ((a+1)b)^{n+1} - b^{n+1}(a+1)^{n+1} \in Z(R), \quad (6)$$

$$\text{and } (b(a+1))^{n+1} - (a+1)^{n+1} b^{n+1} \in Z(R). \quad (7)$$

By adding (6) and (7) and using the fact that $N^2(R) \subseteq Z(R)$, we get

$$ab^{n+1} + b^{n+1}a - (n+1)b^{n+1}a - (n+1)ab^{n+1} \in Z(R).$$

$$\text{That is, } ab^{n+1} + b^{n+1}a - nb^{n+1}a - b^{n+1}a - nab^{n+1} - ab^{n+1} \in Z(R).$$

$$\text{So, } -n[a, b^{n+1}] \in Z(R).$$

$$\text{That is, } n[a, b^{n+1}] \in Z(R).$$

Since R is n -torsion free, we get

$$[a, b^{n+1}] \in Z(R), \text{ where } a \text{ in } N(R), b \text{ in } R. \quad (8)$$

$$\text{Therefore } [a, b^{n+1}] = [a, b]b^n + b[a, b^n] \in Z(R). \quad (9)$$

But by Lemma 2(i), $[a, b^n] = 0$ and hence by (6), we get

$$[a, b]b^n \in Z(R). \quad (10)$$

Thus $[[a, b]b^n, b] = 0 = [[a, b], b]b^n$. By replacing b by $b + 1$ in the above argument and using Lemma 1, we see that

$$[[a, b], b] = 0, a \text{ in } N(R), b \text{ in } R. \quad (11)$$

By using Lemma 2(i), (10) and Lemma 1, we get

$$0 = [a, b^n] = nb^{n-1}[a, b].$$

$$\text{Since } R \text{ is } n\text{-torsion free, we conclude that } b^{n-1}[a, b] = 0. \quad (12)$$

By replacing b by $b + 1$ in (12) and using Lemma 1, we get $[a, b] = 0$, for all a in $N(R)$, b in R . Thus the nilpotent elements are central and hence

$$[x, y] \in Z(R), \text{ for all } x, y \text{ in } R. \quad (13)$$

By using (13) and Lemma 1, we get $0 = [x^n, y^n] = nx^{n-1}[x, y^n]$. By using Lemma 1 and n -torsion free condition, we get $[x, y^n] = 0$, for all x, y in R .

Similarly, $0 = [x, y^n] = ny^{n-1}[x, y]$. This shows that $[x, y] = 0$, for all x, y in R .

Theorem: 4.1.2 Let n be a fixed positive integer and R be an n -torsion free alternative periodic ring with identity such that

$$(xy)^{n+1} - (yx)^{n+1} \in Z(R), \text{ for all } x, y \text{ in } R. \quad (14)$$

If $N(R)$ is commutative, then R is commutative.

Proof: By Lemma 3, $N(R)$ is an ideal of R . Also since $N(R)$ is commutative, $N^2(R) \subseteq Z(R)$. Let a in $N(R)$, b in R and replacing $x = (1 + a)b$ and $y = (1 + a)^{-1}$ in (14), we get

$$((1 + a)b(1 + a)^{-1})^{n+1} - ((1 + a)^{-1}(1 + a)b)^{n+1} \in Z(R).$$

$$\text{That is, } ((1 + a)b(1 + a)^{-1})^{n+1} - b^{n+1} \in Z(R).$$

$$\text{So, } (1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1} \in Z(R). \quad (15)$$

$$\text{Hence } [(1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1}](1 + a) = (1 + a)[(1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1}]$$

$$\text{So, } (1 + a)b^{n+1} - b^{n+1}(1 + a) = (1 + a)[(1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1}].$$

$$\text{That is, } ab^{n+1} - b^{n+1}a = (1 + a)[(1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1}]. \quad (16)$$

$$\text{Since } N(R) \text{ is a commutator ideal, } (1 + a)(ab^{n+1} - b^{n+1}a) = ab^{n+1} - b^{n+1}a.$$

By using (16), we get

$$(1 + a)(ab^{n+1} - b^{n+1}a) = (1 + a)[(1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1}]. \text{ Further since } a \text{ in } N(R), 1 + a \text{ is a unit in } R \text{ and thus } (ab^{n+1} - b^{n+1}a) = [(1 + a)b^{n+1}(1 + a)^{-1} - b^{n+1}].$$

$$\text{Thus } [a, b^{n+1}] \in Z(R), \text{ for all } a \text{ in } N(R) \text{ and } b \text{ in } R. \quad (17)$$

Now suppose that $x_1, x_2, \dots, x_k \in R$. Since $R/Z(R)$ is commutative,

$$(x_1 \cdot x_2 \dots x_k)^{n+1} - x_1^{n+1} \cdot x_2^{n+1} \dots x_k^{n+1} \in Z(R) \subseteq N(R). \text{ Since } N(R) \text{ is commutative, hence}$$

$$[a, (x_1 \cdot x_2 \dots x_k)^{n+1}] = [a, x_1^{n+1} \cdot x_2^{n+1} \dots x_k^{n+1}], \text{ for } a \text{ in } N(R). \quad (18)$$

By using (17) and (18), we conclude that

$$[a, x_1^{n+1} \cdot x_2^{n+1} \dots x_k^{n+1}] \in Z(R), \text{ for } a \text{ in } N(R), x_1, x_2, \dots, x_k \in R. \quad (19)$$

Let S be the subring of R generated by the $(n + 1)^{th}$ powers of elements of R , then by using (19), we get $[a, x] \in Z(S)$, for all a in $N(S)$, x in S . (20)

Here $Z(S)$ and $N(S)$ denote the center of S and the set of nilpotents of S respectively.

Using the fact that S is periodic, $N(S)$ is commutative and using (19), Theorem of [1] shows that S is commutative.

Hence $[x^n, y^n] = 0$, for all x, y in R . (21)

Since R is n -torsion free ring with identity satisfying (21) and the hypothesis $(xy)^{n+1} - (yx)^{n+1}$ is always central. Hence by Theorem 1[3], R is commutative.

We give certain examples which show that all the hypothesis of Theorem 1 and Theorem 2 are essential.

Example: 1 Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(4) \right\}$, and let $n = 5$. Then R satisfies all the hypothesis of Theorem 2 except that R is not n -torsion free. Also R is not commutative. Hence the hypothesis R is n -torsion free cannot be omitted in Theorem 2.

Example: 2 Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(2) \right\}$, and let $n = 2$. This ring shows that the condition n -torsion free cannot be omitted.

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