International Research Journal of Pure Algebra -4(1), 2014, 375-378
Available online through www.rjpa.info ISSN 2248–9037

# COMMUTATIVITY OF ALTERNATIVE PERIODIC RINGS WITH $(xy)^{n+1} - (yx)^{n+1} \in Z(R)$

**Y. S. Kalyan Chakravarthy* & K. Suvarna**

*Department of Mathematics, S. K. University, Ananthapuramu – 515001, India.*

*(Received on: 30-12-13; Revised & Accepted on: 16-01-14)*

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***ABSTRACT***

*Let R be an Alternative periodic ring. In this paper, we show that an $n$-torsion free alternative ring $R$ with identity satisfying $x^n y^n = y^n x^n$ and$(xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R)$ is commutative. We also prove that if $R$ is an $n$-torsion free alternative periodic ring suchthat$(xy)^{n+1} - (yx)^{n+1} \in Z(R)$, $n$ is a fixed positive integer and $N(R)$ is commutative, then R is commutative.*

***AMS Mathematics subject classification**: 17.*

***Key words**: Alternative rings, Periodic rings, Center.*

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## INTRODUCTION

Abu-Khuzam, Bell and Yaqub [2] proved that an $n$-torsion free ring $R$ with identity such that, for all $x, y$ in $R$, $x^n y^n = y^n x^n$ and $(xy)^{n+1} - x^{n+1}y^{n+1} \in Z(R)$ must be commutative. They also have shown that a periodic $n$-torison free ring (not necessarily with identity) for which $(xy)^n - (yx)^n \in Z(R)$ is commutative provided that the nilpotents of $R$ form a commutative set. In this paper, we show that an $n$-torsion free alternative ring $R$ with identity satisfying $x^n y^n = y^n x^n$ and $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z(R)$ is commutative. We also prove that if $R$ is an $n$-torsion free alternative periodic ring suchthat$(xy)^{n+1} - (yx)^{n+1} \in Z(R)$, $n$ is a fixed positive integer and $N(R)$ is commutative, then R is commutative.

## PRELIMINARIESA

Throughout this paper, $R$ denotes an alternative periodic ring, $N(R)$ denotes the set of all nilpotent elements of R, $Z(R)$ the center of $R$, $C(R)$ the commutator ideal of $R$.

In order to prove our main results, we state the following well-known results.

**Lemma: 1[6]** Suppose that $R$ is a ring with identity 1. If $x^m[x,y] = 0$ and $(x+1)^m[x,y] = 0$, for some $x, y$ in $R$ and some integer m>0, then $[x,y] = 0$. A similar statement holds if we assume $[x,y]x^m = 0$ and $[x,y](x+1)^m = 0$ instead.

**Lemma: 2[4]** Let $R$ be an $n$-torsion free ring with identity 1 such that $[x^n, y^n] = 0$ for all $x, y$ in $R$. Let $N(R)$ denote the set of nilpotent elements of $R$. Then
(i) $a \in N(R), x \in R$ *imply* $[a, x^n] = 0$,
(ii) $a \in N(R), b \in N(R)$ *imply* $[a,b] = 0$.

**Lemma: 3** Let $R$ be a periodic ring such that $N(R)$ is commutative. Then the commutator ideal of $R$ is nil and $N(R)$ forms an ideal of $R$.

**Proof:** Let $N(R)$ denote the set of nilpotent elements of $R$ and we use the standard argument for the commutative case to show that $u_1 - u_2 \in N(R)$, whenever $u_1, u_2 \in N(R)$.

We prove this by induction on $k$. That is, if $u_k = 0$ and $r \in R$ then it is true that $(ur)^k = (ru)^k = 0$.

Consider the case $k = 2$.

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***Corresponding author: Y. S. Kalyan Chakravarthy***
**Department of Mathematics, S. K. University, Ananthapuramu – 515001, India.**

Let $u$ be an element of R such that $u^2 = 0$ and let $j$ be a positive integer for which $(ur)^j = e$ is idempotent (possibly zero). Then $re - ere$ is nilpotent and hence commutes with $u$.

That is, $ur(ur)^j - u(ur)^j r(ur)^j = r(ur)^j u - (ur)^j r(ur)^j u.$ (1)

and multiplying on the right of (1) by $u$ gives $ur(ur)^j u = 0$.

So that $(ur)^{j+2} = (ru)^{j+2} = 0$. It follows that u commutes with both ur and ru.

So $(ur)^2 = (ru)^2 = 0$. Now suppose that the result holds for all y with $y^m = 0, m < k$.

Suppose $u^k = 0, k \geq 3$. Determining j as above and multiplying (1) by $r$ on the left and $u$ on the right, we get

$(ru)^{j+2} = ru^2 s + tu^2$ (2)

where $s$ and $t$ are elements of the subring generated by $r$ and $u$. Since $(u^2)^{k-1} = 0$, the inductive hypothesis implies both $ru^2 s$ and $tu^2$ are nilpotent. Therefore (2) shows that $ru$ and $ur$ are nilpotent. Again invoking the fact that $u$ must commute with $ur$ and $ru$, we see that $(ur)^k = (ru)^k = 0$. Hence $N(R)$ is an ideal.

## MAIN RESULTS

**Theorem: 1** Let $R$ be an alternative with identity and $n$ be a fixed positive integer. Suppose that $R$ is $n$-torsion free and that for all $x, y$ in $R$,

$x^n y^n = y^n x^n,$ (3)

and $(xy)^{n+1} - y^{n+1} x^{n+1} \in Z(R).$ (4)

Then $R$ is commutative.

**Proof:** By (3), we have $[x^n, y^n] = 0$ for all $x, y$ in $R$ and by [5], the commutator ideal is nil. This implies that the set of nilpotent elements $N(R)$ forms an ideal.

Hence by Lemma 2(ii), $N(R)$ is a commutative ideal.

This implies that $N^2(R) \subseteq Z(R)$ . (5)

By (4), we have $((a+1)b)^{n+1} - b^{n+1}(a+1)^{n+1} \in Z(R),$ (6)

and $(b(a+1))^{n+1} - (a+1)^{n+1} b^{n+1} \in Z(R).$ (7)

By adding (6) and (7) and using the fact that $N^2(R) \subseteq Z(R)$, we get

$ab^{n+1} + b^{n+1}a - (n+1)b^{n+1}a - (n+1)ab^{n+1} \in Z(R).$

That is, $ab^{n+1} + b^{n+1}a - nb^{n+1}a - b^{n+1}a - nab^{n+1} - ab^{n+1} \in Z(R).$

So, $-n[a, b^{n+1}] \in Z(R).$

That is, $n[a, b^{n+1}] \in Z(R).$

Since $R$ is $n$-torsion free, we get

$[a, b^{n+1}] \in Z(R)$, where $a$ in $N(R)$, $b$ in $R$. (8)

Therefore $[a, b^{n+1}] = [a, b]b^n + b[a, b^n] \in Z(R).$ (9)

But by Lemma 2(i), $[a, b^n] = 0$ and hence by (6), we get

$[a, b]b^n \in Z(R).$ (10)

Thus $[[a,b]b^n, b] = 0 = [[a,b],b]b^n$. By replacing $b$ by $b+1$ in the above argument and using Lemma 1, we see that

$[[a,b],b] = 0$, $a$ in $N(R)$, $b$ in $R$ . (11)

By using Lemma 2(i), (10) and Lemma 1, we get

$0 = [a, b^n] = nb^{n-1}[a,b]$.

Since $R$ is $n$-torsion free, we conclude that $b^{n-1}[a,b] = 0$ . (12)

By replacing $b$ by $b+1$ in (12) and using Lemma 1, we get $[a,b] = 0$, for all $a$ in $N(R)$, $b$ in $R$. Thus the nilpotent elements are central and hence

$[x,y] \in Z(R)$, for all $x, y$ in $R$. (13)

By using (13) and Lemma 1, we get $0 = [x^n, y^n] = nx^{n-1}[x, y^n]$. By using Lemma 1 and $n$-torsion free condition, we get$[x, y^n] = 0$, for all $x, y$ in $R$.]

Similarly, $0 = [x, y^n] = ny^{n-1}[x,y]$. This shows that$[x,y] = 0$, for all $x, y$ in $R$.

**Theorem: 4.1.2** Let $n$ be a fixed positive integer and $R$ be an $n$-torsion free alternative periodic ring with identity such that
$(xy)^{n+1} - (yx)^{n+1} \in Z(R)$, for all $x, y$ in $R$. (14)

If $N(R)$ is commutative, then $R$ is commutative.

**Proof:** By Lemma 3, $N(R)$ is an ideal of $R$. Also since $N(R)$ is commutative, $N^2(R) \subseteq Z(R)$. Let $a$ in $N(R)$, $b$ in $R$ and replacing $x = (1+a)b$ and $y = (1+a)^{-1}$ in (14), we get

$((1+a)b(1+a)^{-1})^{n+1} - ((1+a)^{-1}(1+a)b)^{n+1} \in Z(R)$.

That is,$((1+a)b(1+a)^{-1})^{n+1} - b^{n+1} \in Z(R)$.

So, $(1+a)b^{n+1}(1+a)^{-1} - b^{n+1} \in Z(R)$. (15)

Hence $[(1+a)b^{n+1}(1+a)^{-1} - b^{n+1}](1+a) = (1+a)\left[(1+a)b^{n+1}(1+a)^{-1} - b^{n+1}\right]$

So, $(1+a)b^{n+1} - b^{n+1}(1+a) = (1+a)[(1+a)b^{n+1}(1+a)^{-1} - b^{n+1}]$.

That is, $ab^{n+1} - b^{n+1}a = (1+a)[(1+a)b^{n+1}(1+a)^{-1} - b^{n+1}]$. (16)

Since $N(R)$ is a commutator ideal, $(1+a)(ab^{n+1} - b^{n+1}a) = ab^{n+1} - b^{n+1}a$.

By using (16), we get

$(1+a)(ab^{n+1} - b^{n+1}a) = (1+a)[(1+a)b^{n+1}(1+a)^{-1} - b^{n+1}]$. Further since $a$ in $N(R)$, $1+a$ is a unit in $R$ and thus $(ab^{n+1} - b^{n+1}a) = [(1+a)b^{n+1}(1+a)^{-1} - b^{n+1}]$.

Thus $[a, b^{n+1}] \in Z(R)$, for all $a$ in $N(R)$ and $b$ in $R$. (17)

Now suppose that $x_1, x_2, \dots\dots, x_k \in R$. Since $R/Z(R)$ is commutativite,

$(x_1.x_2 \dots\dots x_k)^{n+1} - x_1^{n+1}.x_2^{n+1} \dots\dots x_k^{n+1} \in Z(R) \subseteq N(R)$. Since $N(R)$ is commutative, hence

$[a, (x_1.x_2 \dots\dots x_k)^{n+1}] = [a, x_1^{n+1}.x_2^{n+1} \dots\dots x_k^{n+1}]$, for a in $N(R)$. (18)

By using (17) and (18), we conclude that

$[a, x_1^{n+1}.x_2^{n+1} \dots\dots x_k^{n+1}] \in Z(R)$, for $a$ in $N(R)$, $x_1, x_2, \dots\dots, x_k \in R$. (19)

 

Let $S$ be the subring of $R$ generated by the $(n+1)^{th}$ powers of elements of $R$, then by using (19), we get $[a, x] \in Z(S)$, for all $a$ in $N(S)$, $x$ in $S$. (20)

Here $Z(S)$ and $N(S)$ denote the center of $S$ and the set of nilpotents of $S$ respectively.

Using the fact that $S$ is periodic, $N(S)$ is commutative and using (19), Theorem of [1] shows that $S$ is commutative.

Hence $[x^n, y^n] = 0$, for all $x, y$ in $R$. (21)

Since $R$ is n-torsion free ring with identity satisfying (21) and the hypothesis $(xy)^{n+1} - (yx)^{n+1}$ is always central. Hence by Theorem 1[3], $R$ is commutative.

We give certain examples which show that all the hypothesis of Theorem 1 and Theorem 2 are essential.

**Example: 1** Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(4) \right\}$, and let $n = 5$. Then $R$ satisfies all the hypothesis of Theorem 2 except that R in not $n$-torsion free. Also $R$ is not commutative. Hence the hypothesis R in $n$-torsion free cannot be omitted in Theorem 2.

**Example: 2** Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(2) \right\}$, and let $n = 2$. This ring shows that the condition $n$-torsion free cannot be omit

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**Source of Support: Nil, Conflict of interest: None Declared**