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ON THE STRUCTURE OF FINITE SIMPLE GROUPS WITH A GIVEN PROPERTY

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ABSTRACT

Let G be a finite group and let x^g denotes the conjugacy class of an element x of G. In this paper we classify all finite simple groups G that union of any four distinct non-trivial conjugacy classes of it together with 1 is a subgroup of G.

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Key words: *Conjugacy class; Simple group; Rational group; EPO-group.*

1. INTRODUCTION

Throughout this paper, G denotes a finite group. The influence of arithmetic structure of conjugacy classes of G, like conjugacy class sizes, the number of conjugacy classes or the number of conjugcy class sizes, on the structure of G is an extensively studied question in group theory. Many authors have studied the influence of some other kind of behaviours of conjugacy classes on the structure of the group. For more details, reader can see the excellent survey article by Camina and Camina [1]. Gumber and Kalra in [4] classified all finite groups G with the property that the union of any i distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G, for $1 \le i \le 3$. An element x of G is called a real element if there exists another element g in G such that $x^g = x^{-1}$. A group G is called a rational group if every element x of G is conjugate to x^m, where m is a natural number coprime to $|x|$. A group G is a EPO-group if every non-identity element of G has prime order. By $Re(G)$ we denote the set of all real elements of G and by $\pi(G)$ we denote the set of all the primes dividing the order of G. The purpose of this paper is to classify all finite simple groups G that union of any four distinct non-trivial conjugacy classes of it together with 1 is a subgroup of G.

Theorem 1.1. *A finite simple group G satisfies in the property that union of any four distinct non-trivial conjugacy classes of it together with 1 is a subgroup of G, if and only if G is isomorphic to $\mathbb{Z}_5$ or A_5.*

2. PROOF OF THEOREM 1.1

We first introduce some lemmas and theorems that will be used in the proof of Theorem 1.1.

Lemma 2.1. [3, Lemma 2.4] *Let G be a group.*
(i) If x is a real element of G and $|x^G|$ is odd, then $x^2 = 1$.
(ii) If x is a real element of G, then every power of x is a real element of G.
(iii) The identity is the unique real element of G if and only if $|G|$ is odd.
(iv) If $N \trianglelefteq G$ and $|\frac{G}{N}|$ is odd, then $Re(G) = Re(N)$.

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Proposition 2.2. [5, Proposition 21] *Let G be a rational group with abelian Sylow 2-subgroup G_2. Then:*
(i) G_2 is elementary abelian
(ii) G splits over G' with G_2 as complement.
(iii) G' is a 3-group.

Corollary 2.3. [6, Corollary 1] *If G is rational, then $Z(G)$ is an elementary abelian 2-group.*

Theorem 2.4. [2, Theorem 1.12] *Let G be a finite EPO-group. Then*
(i) G is nilpotent if and only if G is a p-group of exponent p.
(ii) G is solvable and nonnilpotent if and only if G is a Frobenius group with kernel $P \in Syl_p(G)$, with P a p-group of exponent p and complement $Q \in Syl_q(G)$, with $|Q| = q$.
(iii) G is nonsolvable if and only if $G \cong A_5$.

Now we start the proof of Theorem 1.1.

Let G be a finite abelian simple group. Observe that if union of any four distinct non-trivial conjugacy classes of G together with 1 is a subgroup of it, then 5 is a divisor of $|G|$ and so $|G| = 5k$ for some positive integer k. It is easy to see that if $G \cong \mathbb{Z}_5$, then union of any four distinct non-trivial conjugacy classes of G together with 1 is a subgroup of it. Conversely suppose that G is a finite abelian simple group that union of any four distinct non-trivial conjugacy classes of G together with 1 is a subgroup of it. Let $x, y \in G - \{1\}$ be two distinct elements. Then $\{1, x, x^2, x^3, y\}$ is a subgroup of G such that $y = x^4$. If $z \in G - \{1, x, x^2, x^3\}$, then $\{1, x, x^2, x^3, z\}$ is a subgroup of G such that $z = x^4 = y$. Thus G is a cyclic group of order 5.

Now let G be a finite non-abelian simple group. It is easily seen that if G is isomorphic to A_5, then union of any four distinct non-trivial conjugacy classes of A_5 together with 1 is a subgroup of it. For the converse part, we proceed in a number of steps.

Step 1. $|G|$ is divisible by exactly 3 distinct primes.
Suppose $|G|$ is divisible by at least four distinct primes. We show that order of every element of G is a prime. Let $x_1 \in G$ be of order $p_1 l$ and let $x_2, x_3, x_4 \in G$ be of orders p_2, p_3 and p_4 respectively, where p_1, p_2, p_3, p_4 are distinct primes and $l \geq 2$. Then $H = 1 \cup x_1^G \cup x_2^G \cup x_3^G \cup x_4^G$ is a subgroup of G containing x_1 but not x_1^l. This is a contradiction and thus every element of G is of prime order. Let $x \in G$ be arbitrary and let $|x| = p$, a prime. Let $y_1, y_2, y_3 \in G$ be of orders q_1, q_2 and q_3 respectively, where $q_1 \neq q_2 \neq q_3$ are primes different from p. Let $K = 1 \cup x^G \cup y_1^G \cup y_2^G \cup y_3^G$. Then x^m is conjugate to x whenever m is relatively prime to p. Thus G is a rational group with elementary abelian Sylow 2-subgroup and therefore G is a $\{2,3\}$-group by Proposition 2.2, this is a contradiction to $|\pi(G)| \geq 4$. So $|G|$ is divisible by at most 3 distinct primes and since G is a non-abelian simple group, therefore $|G|$ is divisible by exactly 3 distinct primes.

Step 2. $|Z(G)| = 1$.
Assume $|Z(G)| > 1$ and let p be the largest prime dividing $|Z(G)|$. Suppose $p \geq 5$. Let $x \in Z(G)$ be of order p, $y \in G$ be of order q, where q is a prime different from p and let $H = 1 \cup x^G \cup (x^2)^G \cup (x^3)^G \cup y^G$. Then H is a subgroup of G containing x and y but not containing xy, a contradiction. Thus $p = 2$ or 3. If $p = 2$, then $Z(G)$ is a 2-group. Let S be a Sylow 2-subgroup of G containing $Z(G)$. Let $z \in Z(G)$ be of order 2, $w \neq z \in S$ and $t, u \in G$ be of orders p and q, respectively, where $p, q \in \pi(G)$ are odd primes. Then $K = 1 \cup z^G \cup w^G \cup u^G \cup t^G$ is a subgroup of G not containing zu. Thus $S = Z(G)$ is the unique Sylow 2-

subgroup of G. Therefore $|\frac{G}{Z(G)}|$ is odd and by Lemma 2.1, $Re(G) = Re(Z(G))$. Let $L = 1 \cup z^G \cup u^G \cup t^G \cup (uz)^G$. Now $u^{-1} \in L$ is of order q, therefore $u^{-1} \in u^G$. Thus u is a real element of $G - Z(G)$, a contradiction to $Re(G) = Re(Z(G))$. Now let $p = 3$, then $Z(G)$ is a 3-group or a $\{2,3\}$-group. Let $Z(G)$ is a 3-group and $z \in Z(G)$ be of order 3. Since $|G|$ is divisible by three distinct primes, therefore there are $y_1, y_2 \in G$ of orders q_1, q_2 respectively, where $q_1 \neq q_2$ are primes different from 3. Let $K = 1 \cup z^G \cup (z^2)^G \cup y_1^G \cup y_2^G$. Then K is a subgroup of G containing z and y_1 but not containing zy_1, a contradiction. Now let $Z(G)$ is a $\{2,3\}$-group and let $z_1, z_2 \in Z(G)$ be of orders 2 and 3 respectively. Also let $u \in G$ be of order p, where p is a prime different from 2 and 3. Let $H = 1 \cup (z_1)^G \cup (z_2)^G \cup (z_2^2)^G \cup u^G$. Then H is a subgroup of G containing z_1 and u but not containing $z_1 u$, a contradiction. Hence $|Z(G)| = 1$.

Step 3. Order of every element of G is prime, i.e. G is a EPO-group.
Let $x \in G$ be an arbitrary element of order m. If G contains an element of composite order, then we can choose three non-conjugate elements x_1, x_2 and x_3 of orders different from m. Let $H = 1 \cup x^G \cup x_1^G \cup x_2^G \cup x_3^G$. Then x^r is conjugate to x whenever r is relatively prime to m. Thus G is a rational group. By Corollary 2.3, If G is a rational group, then $Z(G)$ is an elementary abelian 2-group, a contradiction. Therefore order of every element of G is prime, i.e. G is a EPO-group.

Now since G is a EPO-group and $|G|$ is divisible by three primes, therefore by Theorem 2.4, G is not nilpotent and G is isomorphic to A_5.

Now the proof of Theorem 1.1 is complete.

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