Research Journal of Pure Algebra -1(4), July - 2011, Page: 88-92
Available online through www.rjpa.info ISSN 2248–9037

ON STRONG NŐRLUND SUMMABILITY OF ORTHOGONAL EXAPANSION

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(Received on: 25-06-11; Accepted on: 13-07-11)

ABSTRACT

In this paper, we shall prove general theorems which contain two theorems on the Strong Nörlund summability of the orthogonal expansion.

In 1965 Sunouchi G. [9] obtained on the strong summability of orthogonal Series .and in 1967 Sunouchi G.,[10] prove the Approximation of Fourier Series and orthogonal Series
.
In this paper, we obtain the comparable result of [9] and [10] with general Strong Nörlund summability of orthogonal expansion.

Key Word: *Strong Nörlund summability, orthogonal Series.*

INTRODUCTION:

Let $\{\phi_n(x)\}$ be an orthonormal system of L^2 -integrable function defined in $[a,b]$ we consider the orthonormal series

$$\sum_{n=0}^{\infty} c_n \phi_n(x) \tag{1}$$

with

$$\sum_{n=0}^{\infty} c_n^2 < \infty . \tag{2}$$

We say the series (1) is (N, p_n)-summable to $s(x)$, if

$$t_n(x) = \frac{1}{p_n} \sum_{k=0}^{\infty} p_{n-k} s_k(x) \to s(x) \text{ as } n \to \infty .$$

Where $\{p_n\}$is a sequence of numbers with $p_0 > 0$ and $p_n \geq 0$ for all n.

It is well known that the method $(\overline{N}, p_n)$ is regular if and only if,

$$\lim_{n\to\infty} \frac{p_n}{p_n} = 0 .$$

Hence, it follows that the method $(\overline{N}, p_n)$ is regular when $\{p_n\} \in M^{\alpha}$

Let

$$S_n = \frac{1}{p_n} \sum_{k=0}^{n} \frac{p_k}{k+1} .$$

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A sequence $\{p_n\}$is said to belong to the class BVM^{α}, if $\{p_n\}\in M^{\alpha}$ and if $\{S_n\}$is a sequence of bounded variation, i.e.

$$\sum_{n=1}^{\infty}\left|S_n - S_{n-1}\right| < \infty .$$

Strong approximation of Cesáro means of order $\alpha > 0$ is obtained by Sunouchi [9],[10], Leindier [3], [4], [5] and Kantawala [1], [2] have discussed the strong approximation of Nőrlund and Euler means of orthogonal series. Sunouchi [9] prove with the strong (C,α)-summability of orthogonal series for two following theorems:

Theorem A: if the orthogonal series (1) and (2) is $(C,1)$-summable to $f(x)$ a.e. in $[a,b]$ for any $\alpha > 0$ and > 0.

Theorem B: if

$$\sum c_m^2 (\log\log m)^2 < \infty .$$

Then, there exists a square integrable function $f(x)$ such that

$$\lim_{n\to\infty}\frac{1}{A_n^{\alpha}}\sum_{v=0}^{n} A_{n-v}^{\alpha-1}\left|s_{n_v}(x) - f(x)\right|^{r} = 0$$

for any $\alpha > 0$ and $r > 0$ a.e. in $[a,b]$and for increasing sequence $\{n_v\}$.

In this paper we shall prove a general theorem on the Strong Nörlund summability of the orthogonal expansion.

Theorem: 1 If the series

$$\sum_{n=1}^{\infty} \{ \sum_{j=1}^{n}\left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^n}{R_{n-1}}\right)^2 \left|a_j\right|^2\}^{1/2}$$

is converges, then the orthogonal expansion

$$\sum_{n=0}^{\infty} a_n \ \Phi_n(x)$$

is summable $\left|n.p_n, q_n\right|$ almost everywhere.

Proof: Let $t_n^{p,q}(x)$ be the nth $(\overline{N}, p_n, q_n)$ mean of series $\sum_{n=0}^{\infty} a_n\phi_n(x)$. Then we have

$$\begin{aligned} t_n^{p,q}(x) &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k s_{k(x)} \\ &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k \sum_{j=0}^{k} a_j\phi_j(x) \\ &= \frac{1}{R_n}\sum_{j=0}^{n} a_j\phi_j(x)\sum_{k=j}^{n} p_{n-k}q_k \\ &= \frac{1}{R_n}\sum_{j=0}^{n} R_n^j a_j\phi_j(x) \end{aligned}$$

where $s_n(x) = \sum_{k=0}^{n} a_k\varphi_k(x)$.

Thus we obtain

$$t_n^{p,q}(x) - t_{n-1}^{p,q}(x) = \frac{1}{R_n}\sum_{j=0}^{n} R_n^j a_j \phi_j(x) - \frac{1}{R_{n-1}}\sum_{j=0}^{n-1} R_{n-1}^j a_j \varphi_j(x)$$

$$= \frac{1}{R_n}\sum_{j=1}^{n} R_n^j a_j \phi_j(x) - \frac{1}{R_{n-1}}\sum_{j=1}^{n-1} R_{n-1}^j a_j \varphi_j(x)$$

$$= \frac{1}{R_n}\sum_{j=1}^{n} R_n^j a_j \phi_j(x) - \frac{1}{R_{n-1}}\sum_{j=1}^{n} R_{n-1}^j a_j \varphi_j(x)$$

$$= \sum_{j=1}^{n}\left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\right) a_j \phi_j(x).$$

Using the Schwarz's inequality and the orthogononality, we obtain

$$\int_a^b |\Delta t_n^{p,q}(x)|\,dx \le (b-a)^{1/2}\left\{\int_a^b |\Delta t_n^{p,q}(x)|^2 dx\right\}^{1/2}$$

$$= (b-a)^{1/2}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\right)^2 |a_j|^2\right\}^{1/2}$$

and therefore

$$\int_a^b |\Delta t_n^{p,q}(x)|\,dx = (b-a)^{1/2}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\right)^2 |a_j|^2\right\}^{1/2}$$

which is convergent by the assumption and from the Beppo-Leni Lemma we complete the proof.

We need the following corollaries from our theorem.

Corollary 1: [6, 7] If the series

$$\sum_{n=1}^{\infty}\frac{P_n}{p_n P_{n-1}}\left\{\sum_{j=1}^{n} P_{n-j}^2\left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_n}\right)^2 |a_j|^2\right\}^{1/2}$$

Converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \phi_n(x)$$

is summable $(\overline{N}, p_n)$ almost everywhere.

Proof: The proof follows from our theorem and the fact that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{P_{n-j}}{P_n} - \frac{P_{n-1-j}}{P_{n-1}}$$

$$= \frac{1}{P_n P_{n-1}}\left(P_{n-1}P_{n-j} - P_n P_{n-1-j}\right)$$

$$= \frac{1}{P_n P_{n-1}}\left\{(P_n - p_n)P_{n-j} - P_n\left(P_{n-j} - p_{n-j}\right)\right\}$$

$$= \frac{1}{P_n P_{n-1}}\left(P_n P_{n-j} - p_n P_{n-j} - P_n P_{n-j} + p_{n-j}P_n\right)$$

$$= \frac{P_n}{P_n P_{n-1}} \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right) p_{n-j} \text{ for all } p_n = 1.$$

Corollary2: [8] If the series

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left\{ \sum_{j=1}^{n} Q_{j-1}^2 a_j^2 \right\}^{1/2}$$

Converges, the the orthogonal series

$$\sum_{n=0}^{n} a_n \phi_n(x)$$

Summable $\left(\overline{N}, p_n\right)$ almost everywhere.

Proof: The proof follows from theorem 1 and the fact that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{Q_n - Q_{j-1}}{Q_n} - \frac{Q_{n-1} - Q_{j-1}}{Q_{n-1}}$$

$$= Q_{j-1} \left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}} \right)$$

$$= - \frac{q_n Q_{j-1}}{Q_n Q_{n-1}} \text{ for all } p_n = 1$$

o r the application of these corollaries, see Okuyama [6,7,8]

If we put

$$\omega(j) = \frac{1}{j} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 .$$

Then we have the following theorem from theorem 1.

Theorem 2. Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |a_n|^2 \Omega(n)\omega(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} a_n \varphi(x)$ is summable $\left|\overline{N}, p_n, q_n\right|$ almost everywhere, where $\omega(n)$ is define by (2).

Proof: We have by Schwarz inequality

$$\sum_{n=1}^{\infty} \int_a^b \left| \Delta t_n^{p,q}(x) \right| \le A \sum_{n=1}^{\infty} \{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \}^{1/2}$$

$$= A \sum_{n=1}^{\infty} \frac{1}{n^{1/2} \Omega(n)^{1/2}} \{ n\Omega(n) \sum_{j=1}^{n} (\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}})^2 |a_j|^2 \}^{1/2}$$

$$\le A \left\{ \sum_{n=1}^{\infty} \frac{1}{n\Omega(n)} \right\}^{1/2} \left\{ \sum_{n=1}^{\infty} n\Omega(n) \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{1/2}$$

$$\leq \text{ A}\left\{\sum_{j=1}^{\infty}\left|a_j\right|^2\right\} \sum_{n=j}^{\infty} n\Omega(n)\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2$$

$$\leq \text{ A}\left\{\sum_{j=1}^{\infty}\left|a_j\right|^2 \frac{\Omega(j)}{j}\sum_{n=j}^{\infty} n^2\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2\right\}^{1/2}$$

$$= \text{ A}\left\{\sum_{j=1}^{\infty}\left|a_j\right|^2 \Omega(j)\omega(j)\right\}^{1/2} < \infty$$

This completes the proof of theorem 2 from the same reason of the proof of theorem 1.

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