International Research Journal of Pure Algebra -4(2), 2014, 393-397
Available online through www.rjpa.info ISSN 2248–9037

COMMON FIXED POINT THEOREM FOR THREE MAPPINGS IN CONE METRIC SPACES

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(Received on: 27-01-14; Revised & Accepted on: 12-02-14)

ABSTRACT

The aim of this paper is to obtain coincidence point and common fixed point for three mappings satisfying generalized contractive conditions in a cone metric spaces. Our results generalize some well-known recent results in the literature.

***Key Words**: common fixed point, coincidence point, cone metric spaces.*

***2010 Mathematics Subject Classification:** 47H10, 54H25.*

1. INTRODUCTION

Huang and Zhang [1] generalized the notion of metric space by replacing the set of real numbers by ordered Banach space and defined cone metric space and extended Banach type fixed point theorems for contractive type mappings. Subsequently, some other authors [4, 5, 6] studied properties of cone metric spaces and fixed point results of mappings satisfying contractive type condition in cone metric spaces. Recently, Stojan Redenovic [3] has obtained coincidence point results for two mappings in cone metric spaces which satisfies new contractive conditions. The same concept was further extended by M. Rangamma and K. Prudhvi [6] and proved coincidence point results and common fixed point results for three maps.

The purpose of this paper to extend and generalize the results of [3] and [6].

First, we recall basic definitions and known results of cone metric spaces that are needed in the sequel.

Definition: 1.1[1] Let E be a real Banach space and P be a subset of E. The set P is called a cone if and only if
i) P is closed, non-empty and $P \neq \{0\}$;
ii) $a, b \in$ R, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P.

The cone P is called normal if there is a number $K \geq 1$ such that for all $x, y \in E$:

$$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\| \quad (1.1.1)$$

The least number $K \geq 1$ satisfying (1.1.1) is called the normal constant of P.

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In the following we always suppose that E is a real Banach space and P is an order cone in E with $int\ P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition: 1.2 Let X be a non-empty set. Suppose that the mapping $d: X \times X \to E$ satisfies
(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example: 1.3 Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\} \subset R^2, X = R$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, \propto |x - y|)$, where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition: 1.4 Let (X, d) be a cone metric space. We say that $\{x_n\}$ is
i) a convergent sequence if for any $0 \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$, for some fixed x in X. we denote this $x_n \to x (n \to \infty)$;
ii) a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N, d(x_n, x_m) \ll c$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ is converges to x if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

Definition: 1.5 Let $f, g: X \to X$. Then the pair (f, g) is said to be (IT)-commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

Definition: 1.6 For the mapping $f, g: X \to X$, let $c(f, g)$ denotes set of coincidence points of f, g that is $c(f, g) = \{z \in X: fz = gz\}$.

Lemma: 1.7[1] Let (X, d) be a cone metric space, and let P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then
i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $(n \to \infty)$.
ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $(n, m \to \infty)$.

2. MAIN RESULT

In this section we obtain coincidence points and common fixed point theorems for three maps in cone metric spaces.

The following theorem extends and improves Theorem 2.1 of [6].

Theorem: 2.1 Let (X, d) be a cone metric space and P a normal cone with normal constant L. Suppose that the self maps E, F, T on X satisfy the condition

$$d(Ex, Fy) \leq c_1 d(Tx, Ty) + c_2 d(Tx, Ex) + c_3 d(Tx, Fy) + c_4 d(Ty, Fy) + c_5 d(Ty, Ex) \quad (2.1.1)$$

where c_1, c_2, c_3, c_4 and c_5 are non-negative reals with $c_1 + c_2 + c_3 + c_4 + c_5 < 1$. If $E(X) \cup F(X) \subseteq T(X)$ and $T(X)$ is a complete subspace of X. Then the maps E,F and T have a coincidence point p in X. Moreover, if (E,T) and (F,T) are (IT)-commuting at p, then E,F and T have a unique common fixed point.

Proof: Suppose x_o is an arbitrary point of X and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = Ex_{2n} = Tx_{2n+1} \ and$$

$$y_{2n+1} = Fx_{2n+1} = Tx_{2n+2}$$

for all $n = 0,1,2, ...$ By (2.1.1), we have

$$d(y_{2n}, y_{2n+1}) = d(Ex_{2n}, Fx_{2n+1})$$

$$\leq c_1 d(Tx_{2n}, Tx_{2n+1}) + c_2 d(Tx_{2n}, Ex_{2n}) + c_3 d(Tx_{2n}, Fx_{2n+1}) + c_4 d(Tx_{2n+1}, Fx_{2n+1}) + c_5 d(Tx_{2n+1}, Ex_{2n})$$

$$\leq c_1 d(Fx_{2n-1}, Ex_{2n}) + c_2 d(Fx_{2n-1}, Ex_{2n}) + c_3 d(Fx_{2n-1}, Fx_{2n+1}) + c_4 d(Ex_{2n}, Fx_{2n+1}) + c_5 d(Ex_{2n}, Ex_{2n})$$

$$\leq c_1 d(Fx_{2n-1}, Ex_{2n}) + c_2 d(Fx_{2n-1}, Ex_{2n}) + c_3[d(Fx_{2n-1}, Ex_{2n}) + d(Ex_{2n}, Fx_{2n+1})] + c_4 d(Ex_{2n}, Fx_{2n+1})$$

$$d(y_{2n}, y_{2n+1}) \leq (c_1 + c_2 + c_3) d(Fx_{2n-1}, Ex_{2n}) + (c_3 + c_4) d(y_{2n}, y_{2n+1})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{(c_1 + c_2 + c_3)}{(1 - c_3 - c_4)} d(Ex_{2n}, Fx_{2n-1})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{(c_1 + c_2 + c_3)}{(1 - c_3 - c_4)} d(y_{2n}, y_{2n-1})$$

$$d(y_{2n}, y_{2n+1}) \leq \lambda d(y_{2n}, y_{2n-1}).$$

Similarly, it can be shown that

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda d(y_{2n}, y_{2n+1})$$

Therefore, for all n,

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \leq \cdots \leq \lambda^{n+1} d(y_0, y_1)$$

Now, for any m>n,

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m)$$

$$\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}] d(y_1, y_0)$$

$$\leq \frac{\lambda^n}{1-\lambda} d(y_1, y_0)$$

From (1.1.1), we have

$$d(y_n, y_m) \leq \frac{\lambda^n}{1-\lambda} K\, d(y_1, y_0).$$

This is $d(y_n, y_m) \to 0$ as $n, m \to \infty$, (since $0 < \lambda < 1$).

Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{Tx_n\}$. Therefore, $\{Tx_n\}$ is a Cauchy sequence. Since $T(X)$ is complete, there exists q in $T(X)$ such that $\{Tx_n\} \to q$ as $n \to \infty$. Consequently, we can find p in X such that $T(p) = q$. We shall show that $Tp = Ep = Fp$.

Note that $d(Tp, Ep) = d(q, Ep)$.

Let us estimate $d(Tp, Ep)$. We have, by the triangle inequality,

$$d(Tp, Ep) \leq d(Tp, Tx_{2n+2}) + d(Tx_{2n+2}, Ep)$$

$$= d(q, Tx_{2n+2}) + d(Ep, Fx_{2n+1}) \quad (2.1.2)$$

By the contractive condition, we get

$$d(Ep, Fx_{2n+1}) \leq c_1 d(Tp, Tx_{2n+1}) + c_2 d(Tp, Ep) + c_3 d(Tp, Fx_{2n+1}) + c_4 d(Tx_{2n+1}, Fx_{2n+1}) + c_5 d(Tx_{2n+1}, Ep)$$

$$\leq c_1 d(Tp, y_{2n}) + c_2 d(Tp, Ep) + c_3 d(Tp, y_{2n+1}) + c_4 d(y_{2n}, y_{2n+1}) + c_5 d(y_{2n}, Ep)$$

$$= c_1 d(q, Tx_{2n+1}) + c_2 d(q, Ep) + c_3 d(q, Tx_{2n+2}) + c_4 d(Tx_{2n+1}, Tx_{2n+2}) + c_5 d(Tx_{2n+1}, Ep)$$

$$\leq c_1 d(q, Tx_{2n+1}) + c_2[d(q, Tx_{2n+1}) + d(Tx_{2n+1}, Ep)] + c_3[d(q, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})] + c_4 d(Tx_{2n+1}, Tx_{2n+2})$$

$d(Ep, Fx_{2n+1}) \leq \frac{(c_1 + c_2 + c_3)}{(1 - c_3 - c_4)} d(q, Tx_{2n+1})$

$d(Ep, Fx_{2n+1}) = \lambda d(q, Tx_{2n+1}) \to 0$ as $n \to \infty$.

Therefore, for large n, and from (2.1.2), we get

$d(Tp, Ep) \leq d(q, Tx_{2n+2}) \leq d(q, q) = 0,$

which leads to $d(Tp, Ep) = 0$ and hence $Tp = q = Ep$ (2.1.3)

Similarly, we can show

$$Tp = q = Fp \tag{2.1.4}$$

From (2.1.3) and (2.1.4), it follows that

$$q = Ep = Fp = Tp \tag{2.1.5}$$

p is a coincidence point of E, F, T.

Since, (E, T), (F, T) are (IT)-commuting at$p$,

We get by (2.1.5) and contraction condition,

$d(EEp, Ep) = d(EEp, Fp) = d(Eq, Fp)$

$\leq c_1 d(Tq, Tp) + c_2 d(Tq, Eq) + c_3 d(Tq, Fp) + c_4 d(Tp, Fp) + c_5 d(Tp, Eq)$

$= c_1 d(Tq, q) + c_2 d(Tq, Eq) + c_3 d(Tq, q) + c_4 d(q, q) + c_5 d(q, Eq)$

$\leq (c_1 + c_2 + c_3) d(Tq, q) + (c_2 + c_5) d(q, Eq)$

$d(EEp, Ep) \leq \frac{(c_1 + c_2 + c_3)}{(1 - c_2 - c_5)} d(Tq, q)$

$\leq \lambda\, d(EEp, Ep)$

a contradiction, (since $\lambda < 1$) and $Ep = Tp$.

Therefore, $EEp = Ep$

$EEp = ETp = TEp$

$\Rightarrow EEp = TEp = Ep = q$

Therefore, $Ep (= q)$ is a common fixed point of E and T. (2.1.6)

Similarly, we get

$Fp = FFp = FTp = TFp$

$\Rightarrow \quad FFp = TFp = Fp = q.$

Therefore, $Fp = Ep (= q)$ is a common fixed point of F and T. (2.1.7)

From (2.1.6) and (2.1.7) it follows that E, F and T have a common fixed point q. The uniqueness of the common fixed point of q follows equation (2.1.1). Indeed, let q_1 be another fixed point of E, F and T. Consider,

$d(q, q_1) = d(Ep, Fp_1)$

$\leq c_1 d(Tp, Tp_1) + c_2 d(Tp, Ep) + c_3 d(Tp, Fp_1) + c_4 d(Tp_1, Fp_1) + c_5 d(Tp_1, Ep)$

$\leq (c_1 + c_2 + c_3) d(Tp, Tp_1) + (c_2 + c_5) d(Tp_1, Ep)$

$$d(Ep, Fp_1) \leq \frac{(c_1 + c_2 + c_3)}{(1 - c_2 - c_5)} d(Tp, Tp_1)$$

$$\leq \lambda d(Ep, Fp_1) = \lambda d(q, q_1)$$

As $0 < \lambda < 1$, it follows that $d(q, q_1) = 0$ that is $q = q_1$.

Therefore, E, F and T have a unique common fixed point.

Remark: 2.2

(i) Taking $E = F, T = I$ (identity map) and $c_2 = c_3 = c_4 = c_5 = 0$ in Theorem 2.1, we obtain the result of Banach [7].

(ii) If $c_1 = c_3 = c_5 = 0$, Theorem 2.1 generalize Theorem 3 of [1], Theorem 2.3 of [4] and Theorem 2.6 of [2].

(iii) If $c_1 = c_2 = c_4 = 0$, Theorem 2.1 generalize Theorem 5 of [1], Theorem 2.5 of [4] and Theorem 2.7 of [2].

ACKNOWLEDGEMENTS

The authors sincerely thanks Prof. B.N. Dubey for his valuable suggestions in the preparation of this paper.

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Source of Support: Nil, Conflict of interest: None Declared

