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SOME RESULTS ON THE GROUP INVERSE OF BLOCK MATRIX OVER RIGHT ORE DOMAINS

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ABSTRACT

Suppose R be a right Ore domain with identity 1, and $R^{m\times n}$ denote the set of all $m\times n$ matrices over R. In this paper, we give the existences and the representations of the group inverse for block matrix $\begin{pmatrix} AC & B \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} BA & B \\ C & 0 \end{pmatrix}$ under the special condition over right Ore domains. The paper's results generalize some relative results of Wang and Fan (International Research Journal of Pure Algebra.3 (12):347-351, 2013)

Keyword: *group inverse; block matrix; right Ore domain.*

1. INTRODUCTION

A square matrix G is said to be group inverse of A, if G satisfies $AGA = A$, $GAG = G$ and $AG = GA$. It is well known that if G exists, it is unique. We then write $G = A^{\#}$. When $A^{\#}$ exists, we denote $A^{\pi} = I - AA^{\#}$.

The Drazin inverses and group inverses of 2×2 block matrices have applications in many areas, especially in singular differential and difference equations and finite Markov chains (see [4-8]). It is important to study them in a larger ring. In 2001, Cao [11] studied the problem over a division ring. Zhang and Bu [1] made a research over a right Ore domain in 2012. The purpose of this paper extends the results of group inverse over skew fields given in [10] to right Ore domains.

A ring is called a right Ore domain if it possesses no zero divisors and every two elements of the ring have a right common multiple. A left Ore domain is defined similarly. Every right(left) Ore domain R can be embedded in the skew field(denoted by K_R) of quotients of itself. For any matrix A over R, the rank of A (denoted by $r(A)$) is defined as the rank of A over K_R (see [1]-[3]).

Let R be a right Ore domain with identity 1, $R^{m\times n}$ be the set of all $m\times n$ matrices over R. The rank of a matrix $A\in R^{m\times n}$ (denoted by $r(A)$) is defined as the rank of A over K_R, i.e., the maximum order of all invertible subblocks of A over K_R. A matrix A is called regular if there exists a matrix X such that $AXA = A$, then X is called a {1}-inverse or regular inverse of A. In this case, denote the set of all {1}-inverses of A by $A\{1\}$. Let $A^{(1)}$ be any {1}-inverse of A. Let $A\in R^{m\times n}$, denote the range and the row range of a matrix A by $R(A)$ and $R_r(A)$, where $R(A) = \{Ax \mid x\in R^{n\times 1}\}$ and $R_r(A) = \{yA \mid y\in R^{1\times m}\}$.

2. SOME LEMMAS

In this section, we give some lemmas which play important role throughout this paper.

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Lemmas: 1[9] Let $A \in R^{n\times n}$, the followings are equivalent:
(i) $A^{\#}$ exists;
(ii) $A^2X = A$ for some $X \in R^{n\times n}$. In this case, $A^{\#} = AX^2$;
(iii) $YA^2 = A$ for some $Y \in R^{n\times n}$. In this case, $A^{\#} = Y^2A$;
(iv) $R(A) = R(A^2)$;
(v) $R_r(A) = R_r(A^2)$.

Lemmas: 2 Let $A, B \in R^{n\times n}$. If AB and BA are all regular, $r(A) = r(B) = r(AB) = r(BA)$, then $(AB)^{\#}$ and $(BA)^{\#}$ exist.

Proof: If $r(A) = r(B) = r(AB) = r(BA)$, then there exist matrices X, Y, Z and W over K_R such that $A = ABX = YBA$ and $B = BAZ = WAB$. Thus $AB(AB)^{(1)}A = ABX = A$, $A(BA)^{(1)}BA = YBA = A,$ $BA(BA)^{(1)}B = BAZ = B,$ $B(AB)^{(1)}AB = WAB = B.$ Thus $R(A) = R(AB),$ $R(B) = R(BA),$ $R_r(A) = R_r(BA)$, $R_r(B) = R_r(AB)$,
Therefore $R(AB) = AR(B) = AR(BA) = ABR(A) = ABR(AB) = R(ABAB)$,
$$R_r(BA) = R_r(B)A = R_r(AB)A = R_r(A)BA = R_r(BA)BA = R_r(BABA).$$

By Lemma 1 we conclude that $(AB)^{\#}$ and $(BA)^{\#}$ both exist.

Lemmas: 3 Let $A \in R^{n\times m}$, $B \in R^{m\times n}$, $(AB)^{\#}$ and $(BA)^{\#}$ exist. then
(i) $(AB)^{\#} = A[(BA)^{\#}]^2 B$;
(ii) $(AB)^{\#} A = A(BA)^{\#}$;
(iii) $B(AB)^{\#} A = BA(BA)^{\#}$;
(iv) $AB(AB)^{\#} A = A$;
(v) $A(BA)^{\#} BA = A$;
(vi) $BA(BA)^{\#} B = B$.

Proof: From Lemma 2.3 of [9], they are obvious.

3. MAIN RESULTS

Theorem: 1 Let $M = \begin{pmatrix} AC & B \\ C & 0 \end{pmatrix} \in R^{2n\times 2n}$, and $r(B) \geq r(C)$, then
(1) $M^{\#}$ exists if and only if $r(B) = r(C) = r(BC) = r(CB)$ and BC, CB are both regular.
(2) If $M^{\#}$ exists, then $M^{\#} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, where
$$M_1 = (BC)^{\pi} AC(BC)^{\#},$$
$$M_2 = (BC)^{\#} B - (BC)^{\pi}[AC(BC)^{\#}]^2 B,$$
$$M_3 = C(BC)^{\#},$$
$$M_4 = -C(BC)^{\#} AC(BC)^{\#} B.$$

Proof: (1): "Only if" part.
If M has a group inverse, by Lemma 1, there exist matrices X and Y over R such that $M = M^2X = YM^2$. Since
$$M^2 = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} BC & 0 \\ CAC & CB \end{pmatrix},$$

$$M=\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

Let

$$X=\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

$$Y=\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix},$$

Then

$$\begin{pmatrix} BC & 0 \\ CAC & CB \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}=\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}\begin{pmatrix} BC & 0 \\ CAC & CB \end{pmatrix}=\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

i.e.,

$$BCX_1=0, \tag{1}$$

$$BCX_2=B, \tag{2}$$

$$CACX_1+CBX_3=C, \tag{3}$$

$$CACX_2+CBX_4=0, \tag{4}$$

$$Y_1BC+Y_2CAC=B, \tag{5}$$

$$Y_2CB=B, \tag{6}$$

$$Y_3BC+Y_4CAC=C, \tag{7}$$

$$Y_4CB=0. \tag{8}$$

Note that $M^{\#}$ exists, it is easy to know B is regular. Using (2) and (6), we have $BC=BB^{(1)}BC=BCX_2B^{(1)}BC$ and $CB=CBB^{(1)}B=CBB^{(1)}Y_2CB$, i.e., both BC and CB are regular.

From (2) and (6) we know $R(B)=R(BC),\ R_r(B)=R_r(CB)$ Therefore, $r(B)=r(BC)=r(CB),$ and $r(C)\geq r(BC)=r(B)\geq r(C)$, i.e., $r(B)=r(C)$.

The "if" part.

By Lemma 2 we know that $(AB)^{\#}$ and $(BA)^{\#}$ both exist. Let $X_1=(BC)^{\pi}B$, $X_2=(BC)^{\#}B$, $X_3=(CB)^{\#}C$, $X_4=-(CB)^{\#}CAC(BC)^{\#}B$. That implies $M=M^2X$ have a solution, so by Lemma 1 $M^{\#}$ exists.

(2): By Lemma 1, the expression of $M^{\#}$ can be got from $M^{\#}=MX^2$. Using Lemma 3, next we can compute that

$$M^{\#}=\begin{pmatrix} AC & B \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 & (BC)^{\#}B \\ (CB)^{\#}C & -(CB)^{\#}CAC(BC)^{\#}B \end{pmatrix}X$$

$$=\begin{pmatrix} B(CB)^{\#}C & AC(BC)^{\#}B-B(CB)^{\#}CAC(BC)^{\#}B \\ 0 & C(BC)^{\#}B \end{pmatrix}\times\begin{pmatrix} 0 & (BC)^{\#}B \\ (CB)^{\#}C & -(CB)^{\#}CAC(BC)^{\#}B \end{pmatrix}$$

$$=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.$$

Corollary: 1 Let $M=\begin{pmatrix} A & B \\ A & 0 \end{pmatrix}\in R^{2n\times 2n}$, and $r(B)\geq r(A)$, then

(1) $M^{\#}$ exists if and only if $r(B)=r(A)=r(BA)=r(AB)$ and BA, AB are both regular.

(2) If $M^{\#}$ exists, then $M^{\#}=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$,

where

$M_1=(BA)^{\pi}A(BA)^{\#}$,

$M_2=(BA)^{\#}B-(BA)^{\pi}[A(BA)^{\#}]^2B$,

$M_3=A(BA)^{\#}$,

$M_4=-A(BA)^{\#}A(BA)^{\#}B$.

Proof: The results is a special case of Theorem 1, let $A=I$ in Theorem, the conclusion is obvious. Similarly, we can get the following results.

Theorem: 2 Let $M=\begin{pmatrix} BA & B \\ C & 0 \end{pmatrix}\in R^{2n\times 2n}$, and $r(B)\leq r(C)$, then

(1) $M^{\#}$ exists if and only if $r(B)=r(C)=r(BC)=r(CB)$ and BC, CB are both regular.

(2) If $M^{\#}$ exists, then $M^{\#}=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$,

where

$M_1=B(CB)^{\#}A-B(CB)^{\#}AB(CB)^{\#}B$,

$M_2=B(CB)^{\#}$,

$M_3=-CB(CB)^{\#}AB(CB)^{\#}A+(CB)^{\#}B-CB[(CB)^{\#}AB]^2(CB)^{\#}B$,

$M_4=-CB(CB)^{\#}AB(CB)^{\#}$.

Corollary: 2 Let $M=\begin{pmatrix} A & A \\ B & 0 \end{pmatrix}\in R^{2n\times 2n}$, and $r(B)\leq r(A)$, then

(1) $M^{\#}$ exists if and only if $r(B)=r(A)=r(BA)=r(AB)$ and BA, AB are both regular.

(2) If $M^{\#}$ exists, then $M^{\#}=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$,

where

$M_1=A(BA)^{\#}-A(BA)^{\#}A(BA)^{\#}A$,

$M_2=A(BA)^{\#}$,

$M_3=-BA(BA)^{\#}A(BA)^{\#}+(BA)^{\#}A-BA[(BA)^{\#}A]^2(BA)^{\#}A$,

$M_4=-BA(BA)^{\#}A(BA)^{\#}$.

Proof: Let $A=I$ in Theorem 2.

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