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GENERALIZED ALPHA GENERALIZED CLOSED SETS IN BITOPOLOGICAL SPACES

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ABSTRACT

In this paper, we introduce generalized alpha generalized closed sets ($g\alpha g$ - closed sets) in bitopological spaces and basic properties of these sets are analyzed. Further we define and study $g\alpha g$ - continuous mappings in bitopological spaces and some of their properties have been investigated.

Keywords: *Bitopological space, ij - $g\alpha g$ - closed set, ij - $g\alpha g$ - open set, ij - $T_{g\alpha g}$ - space, ij - $g\alpha g$ - continuous mappings.*

1. INTRODUCTION

A triple (X, τ_1, τ_2), where X is a non empty set and τ_1, τ_2 are topologies on X is called a bitopological space and J. C. Kelly [2] initiated the study of such spaces . In 1990, M. Jelic [3] introduced the concepts of alpha open sets in bitopological spaces. In 1986, T. Fukutake [6] introduced the concepts of generalized closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces. O. A. El-Tantawy and H. M. Abu-Donia [5] introduced alpha generalized closed sets in bitopological spaces. In 2012, V. Seenivasan and S. Kalaiselvi [7] introduced and studied the concepts of generalized semi generalized closed sets in bitopological spaces.

The purpose of this paper is to introduce a new class of closed sets called generalized alpha generalized closed sets ($g\alpha g$-closed sets) and generalized alpha generalized continuous mappings ($g\alpha g$ - continuous mappings) in bitopological spaces and investigate some of their properties .

2. PRELIMINARIES

Throughout this paper X, Y and Z always represent non empty bitopological spaces (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, ρ_1, ρ_2) on which no separation axioms are assumed unless explicitly mentioned and the integers $i, j, k \in \{1,2\}$.

For a subset A of X τ_i - $cl(A)$ (resp. τ_i - $\text{int}(A)$, τ_i - $\alpha cl(A)$) denote the closure (resp. interior, α - closure) of A with respect to the topology τ_i. By (i, j) we mean the pair of topologies (τ_i, τ_j).

Definition: 2.1 A subset A of a bitopological space (X, τ_1, τ_2) is called
(i) ij - α - open [3] if $A \subseteq \tau_i$ - $\text{int}(\tau_j$ - $cl(\tau_i$ - $\text{int}(A)))$, where $i \neq j; i, j = 1,2$.
(ii) ij - α - closed [3] if $X - A$ is ij - α - open, where $i \neq j; i, j = 1,2$.

Equivalently, a subset A of a bitopological space (X, τ_1, τ_2) is called ij - α - closed if τ_j - $cl(\tau_i$ - $\text{int}(\tau_j$ - $cl(A))) \subseteq A$.

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Definition: 2.2 A subset *A* of a bitopological space (X,τ_1,τ_2) is called

(i) ij - generalized closed (briefly ij - g - closed) [6] if τ_j - $cl(A)\subseteq U$ whenever $A\subseteq U$ and *U* is τ_i - open in *X* .

(ii) ij - generalized open (briefly ij - g - open) [6] if $X-A$ is ij - g - closed.

(iii) ij - α generalized closed (briefly ij - αg - closed) [5] if τ_j - $\alpha cl(A)\subseteq U$ whenever $A\subseteq U$ and *U* is τ_i - open in *X*.

(iv) ij - α generalized open (briefly ij - αg - open) [5] if $X-A$ is ij - αg - closed .

(v) ij - generalized α closed (briefly ij - $g\alpha$ - closed) [4] if τ_j - $\alpha cl(A)\subseteq U$ whenever $A\subseteq U$ and *U* is τ_i - α - open in *X* .

(vi) ij - generalized α open (briefly ij - $g\alpha$ - open) [4] if $X-A$ is ij - $g\alpha$ - closed .

Definition: 2.3 A bitopological space (X,τ_1,τ_2) is called ij - $T_{½}$- space [6] if every ij - g - closed set in it is τ_j - closed.

Definition: 2.4 A map $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is called

(i) τ_j - σ_k - continuous [1] if the inverse image of every σ_k - closed in (Y,σ_1,σ_2) is τ_j - closed in (X,τ_1,τ_2) .

(ii) ij - g - σ_k - continuous [1] if the inverse image of every σ_k - closed in (Y,σ_1,σ_2) is ij - g - closed in (X,τ_1,τ_2) .

(iii) ij - αg - σ_k - continuous if the inverse image of every σ_k - closed in (Y,σ_1,σ_2) is ij - αg - closed in (X,τ_1,τ_2) .

(iv) ij - $g\alpha$ - σ_k - continuous if the inverse image of every σ_k - closed in (Y,σ_1,σ_2) is ij - $g\alpha$ - closed in (X,τ_1,τ_2) .

3. GENERALIZED ALPHA GENERALIZED CLOSED SETS IN BITOPOLOGICAL SPACE

In this section we introduce the concept of ij - $g\alpha g$ - closed sets in bitopological spaces and discuss some of the related properties .

Definition: 3.1 A subset *A* of a bitopological space (X,τ_1,τ_2) is said to be a ij - generalized alpha generalized closed set (briefly ij - $g\alpha g$ - closed) if τ_j - $cl(A)\subseteq U$ whenever $A\subseteq U$ and *U* is τ_i - αg - open in *X*.

Proposition: 3.2 Every τ_j - closed set is ij - $g\alpha g$ - closed set .

Proof: Let *A* be any τ_j - closed set and *U* be any τ_i - αg - open set containing *A*. Then τ_j - $cl(A)=A\subseteq U$. Hence *A* is ij - $g\alpha g$ - closed set.

The converse of the above proposition is not true as seen from the following example.

Example: 3.3 Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\}\}$, and $\tau_2=\{X,\phi,\{a,b\}\}$. Then $\{b,c\}$ is 12 - $g\alpha g$ - closed but not τ_2 - closed.

Proposition: 3.4 Every ij - $g\alpha g$ - closed set is ij - g - closed.

Proof: Let *A* be any ij - $g\alpha g$ - closed set and *U* be any τ_i - open set containing *A*. Since every τ_i - open is τ_i - αg - open set and *A* is ij - $g\alpha g$ - closed set, then τ_j-$cl(A)\subseteq U$. Hence *A* is ij - g - closed set.

The converse of the above proposition is not true as seen from the following example.

Example: 3.5 Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a,b\}\}$, and $\tau_2=\{X,\phi,\{a\},\{b\},\{a,b\}\}$. Then $\{a,b\}$ is 12 - g - closed but not 12 - $g\alpha g$ - closed.

Proposition: 3.6 Every ij - $g\alpha g$ - closed set is ij - αg - closed.

Proof: Let *A* be any ij - $g\alpha g$ - closed set and *U* be any τ_i - open set containing *A*. Since every τ_i - open is τ_i - αg - open set and *A* is ij - $g\alpha g$ - closed set, then τ_j - $\alpha cl(A) \subseteq \tau_j$ - $cl(A) \subseteq U$. Hence *A* is ij - αg - closed set.

The converse of the above proposition is not true as seen from the following example.

Example: 3.7 Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{a\}\}$ and $\tau_2 = \{X,\varphi,\{a\},\{b,c\}\}$. Then $\{a,b\}$ is 12 - αg - closed but not 12 - $g\alpha g$ - closed.

Proposition: 3.8 Every ij - $g\alpha g$ - closed set is ij - $g\alpha$ - closed.

Proof: Let *A* be any ij - $g\alpha g$ - closed set and *U* be any τ_i - α - open set containing *A*.
Then τ_j - $\alpha cl(A) \subseteq \tau_j$ - $cl(A) \subseteq U$. Hence *A* is ij - $g\alpha$ - closed set.

The converse of the above proposition is not true as seen from the following example.

Example: 3.9 Let $X = \{a,b,c,d\}$, $\tau_1 = \{X,\phi,\{a,b,c\}\}$, and $\tau_2 = \{X,\varphi,\{a,d\},\{a,b,d\}\}$.
Then $\{b\}$ is 12 - $g\alpha$ - closed but not 12 - $g\alpha g$ - closed.

Definition: 3.10 A subset *A* of a bitopological space (X,τ_1,τ_2) is said to be a *ij* - generalized alpha generalized open set (briefly ij - $g\alpha g$ - open) if $X - A$ is ij - $g\alpha g$ - closed in (X,τ_1,τ_2) .

Theorem: 3.11 In a bitopological space (X,τ_1,τ_2)

(i) Every τ_j - open set is ij - $g\alpha g$ - open set.

(ii) Every ij - $g\alpha g$ - open set is ij - g - open.

(iii) Every ij - $g\alpha g$ - open set is ij - αg - open and ij - $g\alpha$ - open.

Theorem: 3.12 If *A* and *B* are ij - $g\alpha g$ - closed sets in *X* , then $A \cup B$ is ij - $g\alpha g$ - closed .

Proof: Let *U* be any τ_i - αg - open set containing *A* and *B*. Then $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since *A* and *B* are ij - $g\alpha g$ - closed sets, τ_j - $cl(A) \subseteq U$ and τ_j - $cl(B) \subseteq U$.

Now, τ_j - $cl(A \cup B) = \tau_j$ - $cl(A) \cup \tau_j$ - $cl(B) \subseteq U$ and so τ_j - $cl(A \cup B) \subseteq U$. Hence $A \cup B$ is ij - $g\alpha g$ - closed.

Theorem: 3.13 If a set *A* is ij - $g\alpha g$ - closed, then τ_j - $cl(A) - A$ contains no non empty τ_i - closed set.

Proof: Let *A* be any ij - $g\alpha g$ - closed and *F* be a τ_i - closed set such that $F \subseteq \tau_j$ - $cl(A) - A$. Since *A* is ij - $g\alpha g$ - closed, we have τ_j - $cl(A) \subseteq F^c$. Then $F \subseteq \tau_j$ - $cl(A) \cap (\tau_j$ - $cl(A))^c = \phi$. Hence *F* is empty.

The converse of the above theorem is not true as seen from the following example.

Example: 3.14 Let $X = \{a,b,c\}$, $\tau_1 = \{X,\phi,\{c\}\}$, and $\tau_2 = \{X,\phi,\{a\},\{a,b\}\}$. If $A = \{a\}$, then τ_2 - $cl(A) - A = \{b,c\}$ does not contain non empty τ_1 - closed set. But $A = \{a\}$ is not 12 - $g\alpha g$ - closed.

Theorem: 3.15 A set *A* is ij - $g\alpha g$ - closed if and only if τ_j - $cl(A) - A$ contains no non empty ij - αg - closed set.

Proof: Let *A* be any ij - $g\alpha g$ - closed and *D* be a ij - αg - closed set such that $D \subseteq \tau_j$ - $cl(A) - A$.
Since *A* is ij - $g\alpha g$ - closed, we have τ_j - $cl(A) \subseteq D^c$. Then $D \subseteq \tau_j$ - $cl(A) \cap (\tau_j$ - $cl(A))^c = \phi$.

Thus D is empty.

Conversely, suppose that $\tau_j\text{-}cl(A) - A$ contains no non empty $ij\text{-}\alpha g$-closed set.

Let $A \subseteq G$ and G is $ij\text{-}\alpha g$-open. If $\tau_j\text{-}cl(A) \subseteq G$ then $\tau_j\text{-}cl(A) \cap G^c$ is non empty.

Since $\tau_j\text{-}cl(A)$ is closed and G^c is $ij\text{-}\alpha g$-closed, we have $\tau_j\text{-}cl(A) \cap G^c$ is non empty $ij\text{-}\alpha g$-closed set of $\tau_j\text{-}cl(A) - A$ which is a contradiction. Therefore $\tau_j\text{-}cl(A) \not\subseteq G$. Hence A is $ij\text{-}g\alpha g$-closed.

Theorem: 3.16 If a set A is $ij\text{-}g\alpha g$-closed, then $\tau_i\text{-}cl(\{x\}) \cap A \neq \phi$ holds for each $x \in \tau_j\text{-}cl(A)$.

Proof: If $\tau_i\text{-}cl(\{x\}) \cap A = \phi$ for some $x \in \tau_j\text{-}cl(A)$, then $A \subseteq (\tau_i\text{-}cl(\{x\}))^c$.

Since A is $ij\text{-}g\alpha g$-closed, we have $\tau_j\text{-}cl(A) \subseteq (\tau_i\text{-}cl(\{x\}))^c$. This shows that $x \notin \tau_j\text{-}cl(A)$. This contradicts the assumption.

The converse of the above theorem is not true as seen from the following example.

Example: 3.17 Let $X = \{a,b,c\}$, $\tau_1 = \{X, \phi, \{a\}\}$, and $\tau_2 = \{X, \phi, \{a\}, \{b,c\}\}$. For a subset $A = \{a,b\}$ is not a $12\text{-}g\alpha g$-closed set, but $\tau_1\text{-}cl(\{x\}) \cap A \neq \phi$, for each $x \in \tau_2\text{-}cl(A)$.

Theorem: 3.18 If A is a $ij\text{-}g\alpha g$-closed set of (X, τ_1, τ_2) such that $A \subseteq B \subseteq \tau_j\text{-}cl(A)$, then B is also an $ij\text{-}g\alpha g$-closed of (X, τ_1, τ_2).

Proof: Let U be any $\tau_i\text{-}\alpha g$-open set such that $B \subseteq U$. As A is $ij\text{-}g\alpha g$-closed and $A \subseteq U$, we have $\tau_j\text{-}cl(A) \subseteq U$. Now $B \subseteq \tau_j\text{-}cl(A)$ which gives, $\tau_j\text{-}cl(B) \subseteq \tau_j\text{-}cl(\tau_j\text{-}cl(A)) = \tau_j\text{-}cl(A) \subseteq U$. Thus $\tau_j\text{-}cl(B) \subseteq U$. Hence B is $ij\text{-}g\alpha g$-closed.

Theorem: 3.19 Let $A \subseteq Y \subseteq X$ and suppose that A is $ij\text{-}g\alpha g$-closed in X. Then A is $ij\text{-}g\alpha g$-closed relative to Y.

Theorem: 3.20 If A is $\tau_i\text{-}\alpha g$-open and $ij\text{-}g\alpha g$-closed in X, then A is τ_j-closed in X.

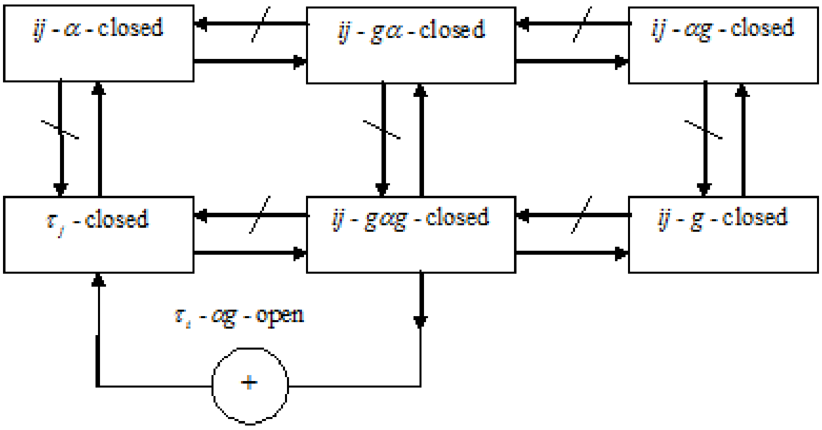
Proof: Since A is $\tau_i\text{-}\alpha g$-open and $ij\text{-}g\alpha g$-closed in X, then $\tau_j\text{-}cl(A) \subseteq A$ and hence A is τ_j-closed in X.

Theorem: 3.21 For each point x of (X, τ_1, τ_2), either a singleton $\{x\}$ is $\tau_i\text{-}\alpha g$-closed or $\{x\}^c$ is $ij\text{-}g\alpha g$-closed in X.

Proof: If set $\{x\}$ is not $\tau_i\text{-}\alpha g$-closed in X, then $\{x\}^c$ is not $\tau_i\text{-}\alpha g$-open in X and the only $\tau_i\text{-}\alpha g$-open set containing $\{x\}^c$ is the space X itself. Then $\tau_j\text{-}cl(\{x\}^c) \subseteq X$ and so $\{x\}^c$ is $ij\text{-}g\alpha g$-closed in X.

Theorem: 3.22 If a subset A of (X, τ_1, τ_2) is $ij\text{-}g\alpha g$-closed in X, then $\tau_j\text{-}cl(A) - A$ is $ij\text{-}g\alpha g$-open set.

Remark: 3.23 The following diagram shows the relations among the different types of weakly closed sets that were studied in this section:



4. GENERALIZED ALPHA GENERALIZED CONTINUOUS MAPPING

In this section we introduce the concept of ij - $g\alpha g$ - continuous mapping bitopological spaces and discuss some of the related properties.

Definition: 4.1 A mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be ij - $g\alpha g$ - σ_k - continuous if the inverse image of every σ_k - closed in *Y* is ij - $g\alpha g$ - closed in *X*.

Theorem: 4.2 If a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij - $g\alpha g$ - σ_k - continuous, then *f* is ij - αg - σ_k - continuous.

Proof: Let *V* be any σ_k - closed in *Y*. Since *f* is ij - $g\alpha g$ - σ_k - continuous, $f^{-1}(V)$ is ij - $g\alpha g$ - closed in *X*. Then by proposition (3.6), $f^{-1}(V)$ is ij - αg - closed in *X*. Hence *f* is ij - αg - σ_k - continuous.

Theorem: 4.3 If a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij - $g\alpha g$ - σ_k - continuous, then *f* is ij - $g\alpha$ - σ_k - continuous.

Theorem: 4.4 If a mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij - $g\alpha g$ - σ_k - continuous if and only if inverse image of each σ_k - open set of *Y* is ij - $g\alpha g$ - open in *X*.

Proof: Let *f* be ij - $g\alpha g$ - σ_k - continuous. If *V* is any σ_k - open set of *Y* then V^c is σ_k - closed in *Y*. Since *f* is ij - $g\alpha g$ - σ_k - continuous, $f^{-1}(V^c)=(f^{-1}(V))^c$ is ij - $g\alpha g$ - closed in *X*. Hence $f^{-1}(V)$ is ij - $g\alpha g$ - open in *X*.

Conversely, let *V* be any σ_k - closed in *Y*. By hypothesis $f^{-1}(V^c)$ is ij - $g\alpha g$ - open in *X*. Then $f^{-1}(V)$ is ij - $g\alpha g$ - closed in *X*. Hence *f* is ij - $g\alpha g$ - σ_k - continuous.

Theorem: 4.5 If $f_1:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij - $g\alpha g$ - σ_k - continuous, $f_2:(Y,\sigma_1,\sigma_2)\to(Z,\rho_1,\rho_2)$ is ij - g - σ_k - continuous and *Y* is ij - $T_{½}$- space . Then $f_2\circ f_1:(X,\tau_1,\tau_2)\to(Z,\rho_1,\rho_2)$ is ij - $g\alpha g$ - σ_k - continuous.

Proof: Let *V* be any ρ_k - closed in Z . Since f_2 is ij - g - σ_k - continuous and *Y* is ij - $T_{½}$- space, $f_2^{-1}(V)$ is σ_j - closed in *Y*. Since f_1 is ij - $g\alpha g$ - σ_k - continuous, $f_1^{-1}(f_2^{-1}(V))$ is ij - $g\alpha g$ - closed in *X*.
Hence $f_2\circ f_1$ is ij - $g\alpha g$ - σ_k - continuous.

Definition: 4.6 A bitopological space (X,τ_1,τ_2) is called a ij - $T_{g\alpha g}$ - space if every ij - $g\alpha g$ - closed set in it is τ_j - closed.

Proposition: 4.7 Every ij - $T_{½}$- space is a ij - $T_{g\alpha g}$ - space.

Proof: Let (X,τ_1,τ_2) be a ij - $T_{½}$- space and let A be a ij - $g\alpha g$ - closed set in X . By proposition (3.4) , A is a ij - g - closed in X . Since X is a ij - $T_{½}$- space, A is τ_j - closed in X. Hence (X,τ_1,τ_2) is a ij - $T_{g\alpha g}$ - space.

Theorem: 4.8 Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a map:

(i) If (X,τ_1,τ_2) is an ij - $T_{½}$- space then f is ij - g - σ_k - continuous if and only if it is ij - $g\alpha g$ - σ_k - continuous.

(ii) If (X,τ_1,τ_2) is an ij - $T_{g\alpha g}$ - space then f is τ_j - σ_k - continuous if and only if it is ij - $g\alpha g$ - σ_k - continuous.

Proof:

(i) Let V be any σ_k - closed in Y . Since f is ij - g - σ_k - continuous, $f^{-1}(V)$ is ij - g - closed in X. By (X,τ_1,τ_2) is an ij - $T_{½}$- space, which implies, $f^{-1}(V)$ is τ_j - closed. By proposition (3.2), $f^{-1}(V)$ is ij - $g\alpha g$ - closed in X. Hence f is ij - $g\alpha g$ - σ_k - continuous.

Conversely, suppose that f is ij - $g\alpha g$ - σ_k - continuous. Let V be any σ_k - closed in Y.

Then $f^{-1}(V)$ is ij - $g\alpha g$ - closed in X. By proposition (3.4), $f^{-1}(V)$ is ij - g - closed in X. Hence f is ij - g - σ_k - continuous.

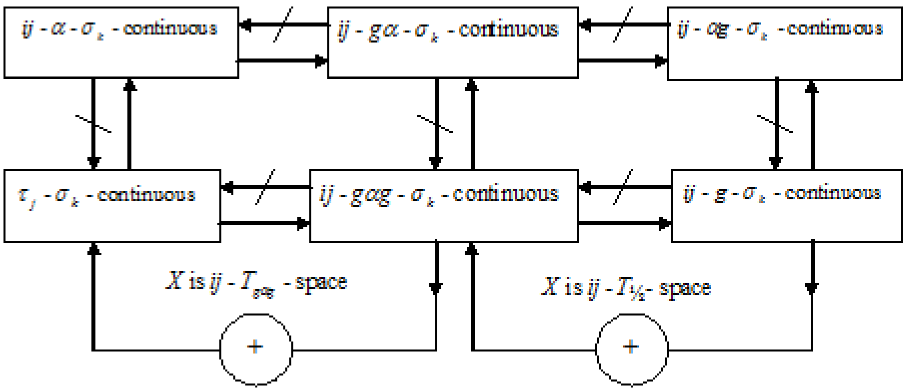
(ii) Let V be any σ_k - closed in Y. Since f is τ_j - σ_k - continuous, $f^{-1}(V)$ is τ_j - closed in X.

By proposition (3.2), $f^{-1}(V)$ is ij - $g\alpha g$ - closed in X. Hence f is ij - $g\alpha g$ - σ_k - continuous.

Conversely, suppose that f is ij - $g\alpha g$ - σ_k - continuous. Let V be any σ_k - closed in Y.

Then $f^{-1}(V)$ is ij - $g\alpha g$ - closed in X. By (X,τ_1,τ_2) is an ij - $T_{g\alpha g}$ - space, which implies, $f^{-1}(V)$ is τ_j - closed in X. Hence f is τ_j - σ_k - continuous.

Remark: 4.9 The following diagram shows the relations among the different types of weakly continuous mappings that were studied in this section:



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