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A COMMON FIXED POINT THEOREM WITH INTEGRAL TYPE INEQUALITY

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ABSTRACT

The purpose of this paper we establish common fixed point theorems for six self maps by using compatible of type (α) with integral type inequality, without appeal to continuity in fuzzy metric space.

Mathematics Subject Classification*: Primary 47H10, Secondary 54H25.*

Keywords: *Compatible maps, Compatible maps of type (α), weakly compatible, common fixed point, Fuzzy metric Space.*

1. INTRODUCTION

In 1965, Zadeh [17] introduced the concept of Fuzzy sets. The concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [9]. George and Veeramani [5] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Grabiec [6] has proved fixed point results for Fuzzy metric space. Singh and Chauhan [13] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Singh and Jain [14] studied the notion of weak compatibility in FM - spaces (introduced by Jungck and Rhoades [7] in metric spaces). Recently, some fixed point results for mappings satisfying an integral type contractive condition are obtained by Altun, Turkoglu and Rhoades [1], Rhoades [11], Vijayaraju, Rhoades and Mohanraj [16] and Sedghi, Shobe and Aliouche [12]. Suzuki [15] showed that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. Jungck *et. al.* [8] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Integral type contraction principle is one of the most popular contraction principles in fixed point theory. The first known result in this direction was given by Branciari [2] in general setting of lebgesgue integral function and proved fixed point theorems in metric spaces. In this paper the results Rangamma and Padma [10] are also assist. Cho [3, 4] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space.

The aim of this paper is to prove a common fixed point theorem for six mappings using compatible of type (α) with integral type inequality, without appeal to continuity.

2. PRELIMINARIES

Definition: 2.1 Let X be any set. A fuzzy set in X is a function with domain X and values in [0, 1].

Definition: 2.2 A binary operation *: [0, 1] × [0, 1]→ [0, 1] is called a continuous t-norm if * is satisfying the following conditions:
a) * is commutative and associative,
b) * is continuous,
c) $a * 1 = a$ for all $a \in [0, 1]$,
d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

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Definition: 2.3 The 3-tuple (X, M, *) is said to be a Fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all x, y, z ∈ X and s, t > 0.
(FM-1) M(x, y, 0) = 0,
(FM-2) M(x, y, t) =1 for all t > 0 if and only if x = y,
(FM-3) M (x, y, t) = M (y, x, t),
(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),
(FM-5) M(x, y, .) : [0, ∞) → [0, 1] is left continuous,
(FM-6) $\lim_{n\to\infty}$ M(x, y, t) =1.
Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0.

Definition: 2.4 Let (X, M,*) be fuzzy metric space then,

a) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for each ε > 0 and each t > 0, there exists n_0 ∈ N such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.
b) A sequence $\{x_n\}$ in X is said to be Cauchy sequence for each ε > 0 and t > 0, there exists n_0 ∈ N such that $M(x_m, x_n, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.
c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition: 2.5 Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be compatible if and only if $M(ASx_n, SAx_n, t) \to 1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition: 2.6 Self map A and S of a fuzzy metric space (X, M, *) are said to be compatible of type (α) if and only if

$$M(ASx_n, SSx_n, t) \to 1 \text{ and } M(AAx_n, ASx_n, t) \to 1 \text{ for all } t > 0,$$

where $\{x_n\}$, is a sequence in X such that Ax_n , $Sx_n \to p$ for some p in X as $n \to \infty$.

Note that compatible map of type (α) is equivalent to the compatible map of type (β).

Lemma: 2.7 In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Lemma: 2.8 Let (X, M, *) be a fuzzy metric space. For all x, y ∈ X, M(x, y, •) is non decreasing.

Lemma: 2.9 Let (X, M, *) be a fuzzy metric space if there exists k ∈ (0, 1) such that M(x, y, kt) ≥ M(x, y, t) then x = y.

Lemma: 2.10 Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number k ∈ (0, 1) such that $M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t)$ for all t > 0 and n ∈ N, then $\{x_n\}$ is a Cauchy sequence in X.

Proposition: 2.11 In a fuzzy metric space (X, M,*), if a* a ≥ a for all a ∈ [0, 1] then a* b = min {a, b} for all a, b∈[0, 1].

Definition 2.12: Self maps A and S of a Fuzzy metric space (X, M, *) are said to be compatible maps of type (β) if $M(AAx_n, SSx_n, t) \to 1$ for all t > 0, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

3. RESULT

Theorem: Let P, Q, A, B, S and T be self maps of a complete fuzzy metric space from X into itself such that
(3.1.1) AB(X) ⊂ P(X), ST(X) ⊂ Q(X),
(3.1.2) AB = BA, ST = TS, QB = BQ, PT = TP,
(3.1.3) (AB, Q) is compatible of type (α) and (ST, P) is weakly compatible
(3.1.4) $\int_0^{M(ABx,STy,kt)} \xi(v)dv \geq \int_0^{\min\{M(ABx,Qx,t),M(STy,Py,t),M(Qx,Py,t),M(Qx,STy,t),M(ABx,Py,t)\}} \xi(v)dv$
for all x, y ∈ X, k ∈ (0, 1), t > 0.

Where $\xi : [0, +\infty] \to [0, +\infty]$ is a lebgesgue integrable mapping which is summable on each compact subset of [0, +∞] non negative and such that for all $\varepsilon > 0, \int_0^{\varepsilon} \xi(v)dv > 0$. Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Let x_0 ∈ X, then by AB(X) ⊂ P(X), there exists a point x_1 ∈ X such that

$$ABx_0 = Px_1,$$

Since ST(X) ⊂ Q(X) for this point x_1, we choose a point x_2 ∈ X such that

$$STx_1=Qx_2.$$

Inductively, we can Now consider a sequence $\{x_n\}$ and $\{y_n\}$ in X as follows

$ABx_{2n} = Px_{2n+1} = y_{2n}$ and $STx_{2n+1} = Qx_{2n+2} = y_{2n+1}$ for n = 1, 2, 3, ...

Step - 1. Put $x = x_{2n+1}$ and $y = x_{2n}$ in (3.1.4), we get

$$\int_0^{M(ABx_{2n+1},STx_{2n},kt)} \xi(v)dv \geq \int_0^{\min\{M(ABx_{2n+1},Qx_{2n+1},t),M(STx_{2n},Px_{2n},t),M(Qx_{2n+1},Px_{2n},t),M(Qx_{2n+1},STx_{2n},t),M(ABx_{2n+1},Px_{2n},t)\}} \xi(v)dv$$

$$\int_0^{M(y_{2n},y_{2n+1},kt)} \xi(v)dv \geq \int_0^{\min\{M(y_{2n-1},y_{2n},t),M(y_{2n-1},y_{2n+1},t),M(y_{2n},y_{2n+1},t)\}} \xi(v)dv$$
$$\geq \int_0^{\min\{M(y_{2n-1},y_{2n},t),M(y_{2n-1},y_{2n},t),M(y_{2n},y_{2n+1},t),M(y_{2n},y_{2n+1},t)\}} \xi(v)dv$$
$$\geq \int_0^{\min\{M(y_{2n-1},y_{2n},t),M(y_{2n},y_{2n+1},t)\}} \xi(v)dv$$

From lemma 2.8 and lemma 2.10, we have

$$\int_0^{M(y_{2n},y_{2n+1},kt)} \xi(v)dv \geq \int_0^{\min\{M(y_{2n-1},y_{2n},t),M(y_{2n},y_{2n+1},t)\}} \xi(v)dv$$

Similarly, we have

$$\int_0^{M(y_{2n+1},y_{2n+2},kt)} \xi(v)dv \geq \int_0^{\min\{M(y_{2n},y_{2n+1},t),M(y_{2n+1},y_{2n+2},t)\}} \xi(v)dv$$

Since $\xi(v)dv$ is a lebesgue integrable function so we have

$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$

Similarly, we have

$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t)$.

Thus, we have

$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t)$ for n = 1, 2, ...

$$\begin{aligned} M(y_n, y_{n+1}, t) &\geq M(y_{n-1}, y_n, \tfrac{t}{k}) \\ &\geq M(y_{n-2}, y_{n-1}, \tfrac{t}{k^2}) \\ &\quad \dots \dots \dots \dots \\ &\geq M(y_0, y_1, \tfrac{t}{k^p}) \to 1 \text{ as } p \to \infty, \text{ and hence } M(y_n, y_{n+1}, t) \to 1 \text{ as } n \to \infty \text{ for any } t > 0. \end{aligned}$$

For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that

$M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$.

For any m, n $\in$ N, we suppose $m \geq n$. Then we have

$$\begin{aligned} M(y_n, y_m, t) &\geq M(y_n, y_{n+1}, \tfrac{t}{m-n}) * M(y_{n+1}, y_{n+2}, \tfrac{t}{m-n}) * \dots * M(y_{m-1}, y_m, \tfrac{t}{m-n}) \\ &\geq (1 - \varepsilon) * (1 - \varepsilon) * \dots * (1 - \varepsilon) \ (m - n) \text{ times} \\ &\geq (1 - \varepsilon) \end{aligned}$$

and hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete $\{y_n\}$ converges to some point $z \in X$. And also subsequences of $\{y_n\}$, they also converges to the same point z i.e.,

$\{Px_{2n+1}\} \to z$ and $\{Qx_{2n}\} \to z$ as $n \to \infty$.

$\{ABx_{2n}\} \to z$ and $\{STx_{2n+1}\} \to z$

As (AB, Q) is compatible of type (α), we have $M(ABABx_{2n}, QQx_{2n}, t) = 1 \ \forall \ t > 0$.

$$ABz = Qz.$$

Step - 2. Put $x = Qx_{2n}$ and $y = x_{2n+1}$ in (3.1.4) we have

$$\int_0^{M(ABQx_{2n},STx_{2n+1},kt)} \xi(v)dv \geq \int_0^{F(M(ABQx_{2n},QQx_{2n},t),M(STx_{2n+1},Px_{2n+1},t),M(QQx_{2n},Px_{2n+1},t),,M(QQx_{2n},STx_{2n+1},t),M(ABQx_{2n},Px_{2n+1},t)\}} \xi(v)dv$$

Taking $n \to \infty$, we get

$$\int_0^{M(ABz,z,kt)} \xi(v)dv \geq \int_0^{F(M(ABz,ABz,t),M(z,z,t),M(ABz,z,t),M(ABz,z,t),M(ABz,z,t)} \xi(v)dv$$

$$\int_0^{M(ABz,z,kt)} \xi(v)dv \geq \int_0^{M(ABz,z,t)} \xi(v)dv$$

Since $\xi(v)dv$ is a lebesgue integrable function this implies

$M(ABz, z, kt) \geq M(ABz, z, t)$

So by lemma **2.10,** we have ABz = z. Therefore ABz = Qz = z.

Step - 3. Put $x = Bz$ and $y = x_{2n+1}$ in (3.1.4) we have

$$\int_0^{M(ABBz,STx_{2n+1},kt)} \xi(v)dv \geq \int_0^{F(M(Px_{2n+1},QBz,t),M(ABBz,Px_{2n+1},t),M(STx_{2n+1},QBz,t),M(ABBz,QBz,t),M(STx_{2n+1},Px_{2n+1},t),} \xi(v)dv$$

Taking $n \to \infty$, we get

$$\int_0^{M(ABBz,z,kt)} \xi(v)dv \geq \int_0^{F(M(z,QBz,t),M(ABBz,z,t),M(z,QBz,t),M(ABBz,QBz,t),M(z,z,t),} \xi(v)dv$$

Since AB = BA and QB = BQ

So AB(Bz) = BA(Bz) = B(ABz) = Bz

And QBz = BQz = Bz.

$$\int_0^{M(Bz,z,kt)} \xi(v)dv \geq \int_0^{F(M(z,Bz,t),M(Bz,z,t),M(z,Bz,t),M(Bz,Bz,t),M(z,z,t),} \xi(v)dv$$

$$\int_0^{M(Bz,z,kt)} \xi(v)dv \geq \int_0^{M(Bz,z,t)} \xi(v)dv$$

Since $\xi(v)dv$ is a lebesgue integrable function this implies

$M(Bz, z, kt) \geq M(Bz, z, t)$

Then by lemma **2.10**, we have Bz = z.

So ABz = z which implies that Az = z.

Hence ABz = Az = Bz = Qz = z.

Step - 4. $AB(X) \subset P(X)$, then their exists $u \in X$ such that ABz = Pu = z.

Put x = z and y = u in (3.1.4) we have

$$\int_0^{M(ABz,STu,kt)} \xi(v)dv \geq \int_0^{F(M(Pu,Qz,t),M(ABz,Pu,t),M(STu,Qz,t),M(ABz,Qz,t),M(STu,Pu,t),} \xi(v)dv$$

$$\int_0^{M(z,STu,kt)} \xi(v)dv \geq \int_0^{F(M(z,z,t),M(z,z,t),M(STu,z,t),M(z,z,t),M(STu,z,t),} \xi(v)dv$$

$\int_0^{M(z,STu,kt)} \xi(v)dv \geq \int_0^{M(z,STu,t)} \xi(v)dv$

Since $\xi(v)dv$ is a lebesgue integrable function this implies

M(STu, z, kt) ≥ M(STu, z, t)

So by lemma **2.10**, we have STu = z. Therefore Pu = STu = z.

Step - 5. (ST, P) is weakly compatible, STPu = PSTu which implies STz = Pz.

Put x = z and y = z in (3.1.4) we have

$\int_0^{M(ABz,STz,kt)} \xi(v)dv \geq \int_0^{F(M(Pz,Qz,t),M(ABz,Pz,t),M(STz,Qz,t),M(ABz,Qz,t),M(STz,Pz,t),} \xi(v)dv$

$\int_0^{M(z,Pz,kt)} \xi(v)dv \geq \int_0^{F(M(Pz,z,t),M(z,Pz,t),M(Pz,z,t),M(z,z,t),M(Pz,Pz,t),} \xi(v)dv$

$\int_0^{M(z,Pz,kt)} \xi(v)dv \geq \int_0^{M(z,Pz,t)} \xi(v)dv$

Since $\xi(v)dv$ is a lebesgue integrable function this implies

M(Pz, z, kt) ≥ M(Pz, z, t)

So by lemma **2.10**, we have Pz = z. So STz = Pz = z.

Step - 6. Put x = z and y = Tz in (3.1.4) we have

$\int_0^{M(ABz,STTz,kt)} \xi(v)dv \geq \int_0^{F(M(PTz,Qz,t),M(ABz,PTz,t),M(STTz,Qz,t),M(ABz,Qz,t),M(STTz,PTz,t),} \xi(v)dv$

$\int_0^{M(z,STTz,kt)} \xi(v)dv \geq \int_0^{F(M(PTz,z,t),M(z,PTz,t),M(STTz,z,t),M(z,z,t),M(STTz,PTz,t),} \xi(v)dv$

Since ST = TS, PT = TP, so we have

PTz = TPz = Tz

And ST(Tz) = TS(Tz) = T(STz) = Tz.

$\int_0^{M(z,Tz,kt)} \xi(v)dv \geq \int_0^{F(M(Tz,z,t),M(z,Tz,t),M(Tz,z,t),M(z,z,t),M(Tz,Tz,t),} \xi(v)dv$

$\int_0^{M(z,Tz,kt)} \xi(v)dv \geq \int_0^{M(z,Tz,t)} \xi(v)dv$

Since $\xi(v)dv$ is a lebesgue integrable function this implies M(Tz, z, kt) ≥ M(Tz, z, t)

So by lemma **2.10**, we have Tz = z. So STz = Sz = z.

Hence STz = Sz = Tz = Pz = z.

On combining we get ABz = Az = Bz = Qz = STz = Sz = Tz = Pz = z. Hence z, is the common fixed point of A, B, S, T, P and Q.

Uniqueness: Let w be the another common fixed point of A, B, S, T, P and Q, then

Step 7. Put x = z and y = w in (3.1.4) we have

$\int_0^{M(ABz,STwkt)} \xi(v)dv \geq \int_0^{F(M(Pw,Qz,t),M(ABz,Pw,t),M(STw,Qz,t),M(ABz,Qz,t),M(STw,Pw,t),} \xi(v)dv$

$\int_0^{M(z,wkt)} \xi(v)dv \geq \int_0^{F(M(w,z,t),M(z,w,t),M(w,z,t),M(z,z,t),M(w,w,t),} \xi(v)dv$

$\int_0^{M(z,w,kt)} \xi(v)dv \geq \int_0^{M(z,w,t)} \xi(v)dv$

Since $\xi(v)dv$ is a lebesgue integrable function this implies

M(w, z, kt) ≥ M(w, z, t)

So by lemma **2.10**, we have w = z.

Hence z, is unique common fixed point of A, B, S, T, P and Q.

If we take B = T = I (Identity mapping) in Theorem 2.1 then we get the following result.

Corollary: Let P, Q, A and S be self maps of a complete fuzzy metric space from X into itself such that

(a) A(X) ⊂ P(X), S(X) ⊂ Q(X),
(b) (A, Q) is compatible of type (β) and (S, P) is weakly compatible

(2.1.4) $\int_0^{M(Bx,Sy,kt)} \xi(v)dv \geq \int_0^{F(M(Py,Qx,t),M(Ax,Py,t),M(Sy,Qx,t),M(Ax,Qx,t),M(Sy,Py,t),} \xi(v)dv$ for all x, y ∈ X, k ∈ (0, 1), t > 0.

where $\xi : [0,+\infty] \to [0,+\infty]$ is a lebgesgue integrable mapping which is summable on each compact subset of [0, +∞] non negative and such that for all $\varepsilon > 0, \int_0^{\varepsilon} \xi(v)dv > 0$. Then A, S, P and Q have a unique common fixed point in X.

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