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ON THE PERFECT NUMBERS AND THEIR ALGEBRAIC MEANING

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ABSTRACT

Let n be a positive integer, n is said to be perfect number if the sum of its positive divisors equals $2n$. We provide characterization of all perfect numbers, and we give an algebraic meaning for what does it mean for a number to be perfect.

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1. INTRODUCTION

A positive integer n is perfect number if n equals the sum of its positive divisors other than it self. All even perfect numbers were characterized by Euler's Theorem. But until now no one knows if there exist an odd perfect number.

In this paper we provide a criterion for all perfect numbers, and then using it to provide an interesting proof of Euler's Theorem for even perfect numbers. Then we relate Number Theory to Algebra by giving an algebraic meaning for perfect numbers, and show how this depends on $\phi(n)$ the Euler phi function.

2. LEMMAS AND THEOREMS

For positive integer n we define $S(n)$ to be the sum of positive divisors of n other than n, that is $S(n) = \sigma(n) - n$. We define $G(n)$ to be the gap between n and $S(n)$, that is $G(n) = n - S(n) = 2n - \sigma(n)$.

Note: $G(n)$ can be negative and that is when n is abundant number.

Lemma: 1 *If* $(n,m)=1$, then $S(nm) = n \cdot S(m) + m \cdot S(n) + S(n)S(m)$.

Proof:
$$\begin{aligned} nm + n \cdot S(m) + m \cdot S(n) + S(n)S(m) &= (n + S(n))(m + S(m)) \\ &= \sigma(n)\sigma(m) = \sigma(nm) \\ &= nm + S(nm). \end{aligned}$$

Lemma: 2 *if* $(n, m) = 1$, *then* $G(nm) = G(n)G(m) - 2\ S(n)S(m)$.

Proof:
$$\begin{aligned} G(n)G(m) &= (n - S(n))(m - S(m)) \\ &= nm - n \cdot S(m) - m \cdot S(n) - S(n)S(m) + 2\ S(n)S(m) \\ &= nm - S(nm) + 2\ S(n)S(m) \\ &= G(nm) + 2\ S(n)S(m). \end{aligned}$$

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Theorem: 1 *If* $(n,m)=1$, *then the number* nm *is perfect iff* $\dfrac{n}{S(n)}=\dfrac{\sigma(m)}{G(m)}$.

Proof: $G(nm)=G(n)G(m)-2\ S(n).S(m)$

$$\begin{aligned}&=(n-S(n))G(m)-2\ S(n)S(m)\\&=n\cdot G(m)-S(n)G(m)-2\ S(n)S(m)\\&=n\cdot G(m)-S(n)\left[G(m)+2\ S(m)\right]\\&=n\cdot G(m)-S(n)\sigma(m).\end{aligned}$$

$$nm\ is\ perfect\ \ iff\ \ G(nm)=0\Leftrightarrow\ \frac{n}{S(n)}=\frac{\sigma(m)}{G(m)}\ .$$

If p prime doesn't divide m then the proof of Theorem 1 allows us to drive the value of $G(p^k m)$ in terms of $G(p^{k-1}m)$ as follows.

Corollary: 1 If p doesn't divide m then $G(p^k m)=p\cdot G(p^{k-1}m)-\sigma(m)$. In particulate $p^k m$ perfect number iff $G(p^{k-1}m)=\dfrac{\sigma(m)}{p}$.

Proof: $G(p^k m)=p^k\cdot G(m)-(p^{k-1}+p^{k-2}+\cdots+p+1)\sigma(m)$

$$\begin{aligned}&=p\left[\,p^{k-1}G(m)-(p^{k-2}+p^{k-3}+\cdots+p+1)\,\sigma(m)\right]-\sigma(m)\\&=p\cdot G(p^{k-1}m)-\sigma(m).\end{aligned}$$

In Particular if $k=1$, $G(pm)=p\cdot G(m)-\sigma(m)$.

Remark: *If* $(n,m)=1$ *then*

$$\frac{n}{S(n)}=\frac{nm}{S(n)m}>\frac{nm}{S(nm)}.$$

This leads to the following corollary.

Corollary: 2 If n composite, then for every prime p divides n, we have $\dfrac{n}{S\,(n)}<p$.

Proof: Let $p^k\,\|\,n$, then

$$\frac{n}{S(n)}<\frac{p^k}{S(p^k)}=\frac{p^k}{\left(p^{k-1}+p^{k-2}+\cdots+p+1\right)}=\left(p-1\right)+\frac{1}{p^{k-1}+p^{k-2}+\cdots+p+1}<p.$$

Now using Theorem 1 we provide an interesting proof to the well known theorem of Euler regarding even perfect numbers.

Theorem: 2 If a is an odd positive integer then $n=2^k\cdot a$ is perfect number iff $\sigma(2^k)=a$, and a is prime.

Proof: Suppose $\sigma(2^k)=a,\ \ a\ \ prime,\ \ then\ \dfrac{\sigma(2^k)}{G(2^k)}=\dfrac{a}{1}=\dfrac{a}{S(a)}$.

Thus $2^k\cdot a$ is perfect .

Conversely, suppose $2^k \cdot a$ is perfect. If a composite, then $\frac{a}{S(a)} < p$, for every prime p divides a. In particular $\frac{a}{S(a)} < p$ for every prime p divides $\sigma\left(2^k\right)$.

Thus $\frac{a}{S(a)} < \sigma(2^k) = \frac{\sigma(2^k)}{G(2^k)}$.

Hence by Theorem 1, $2^k.a$ is not perfect, contradiction. Thus a must be prime, and since $\sigma(2^k)$ divides a, then $\sigma(2^k) = a,\ a$ is prime.

3. THE ALGEBRAIC MEANING OF PERFECT NUMBERS

Let n positive integer, then $(Z_n, +)$ is a cyclic group of order n having $\phi(n)$ generator, and $\tau(n)$ distinct subgroups, where $\phi(n)$ is the Euler phi function, $\tau(n)$ number of positive divisors of n.

We will regard all the subgroups of $(Z_n, +)$ as proper subgroups except the group it self (in particular the subgroup consists of the identity element 0 is proper).

For every divisor d of n there is a subgroup of order d generated by $\frac{n}{d}$.

Thus $S(n)$ sum of positive divisors of n other than n equals the sum of orders of all proper subgroups.

If every x, $0 \le x \le n-1$ were used once in the proper subgroups then the sum of orders of all proper subgroups equals n, and n perfect number, but the fact is some $x's$ doesn't used at all (that is happens when $(x, n) = 1$, that is there are $\phi(n)$ of such $x's$).

On the other hand some $x's$ used more than once (the identity element 0 used in every proper subgroup).

So for an element x in a proper subgroup, we introduce the term ((Extra Used)) of x to mean the number of times it's used in all proper subgroups except for one time, and denoted by $E(x)$.

And for a positive integer n, we introduce the term ((Total Extra)) to mean the sum of all ((Extra Used)) of all elements in all proper subgroups of the group $(Z_n, +)$, and denoted by $T(n)$.

For example: for $n = 28$, the proper subgroups of $(Z_n, +)$ are:
{0}.
{0, 14}.
{0, 7, 14, 21}.
{0, 4, 8, 12, 16, 20, 24}.
{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26}.

Thus we have

$E(0) = 4,\ E(4) = 1,\ E(8) = 1,\ E(12) = 1,\ E(14) = 2,\ E(16) = 1,\ E(20) = 1,\ E(24) = 1.$ and
$E(2) = E(6) = E(7) = E(10) = E(18) = E(21) = E(22) = E(26) = 0$.

Therefore, $T(28) = E(0) + E(4) + E(8) + E(12) + E(14) + E(16) + E(20) + E(24) = 12$.

Theorem: 3 If n positive integer, then n is perfect number iff number of generators of the group $(Z_n, +)$ equals to $T(n)$, where $T(n)$ as defined above.

Proof: Sum of divisors of n other than n, S(n) equals the sum of orders of all proper subgroups of $(Z_n, +)$. That is

$$S(n) = n - \phi(n) + T(n)$$
$$n - S(n) = \phi(n) - T(n)$$
$$G(n) = \phi(n) - T(n).$$

n is perfect number $\Leftrightarrow G(n) = 0 \Leftrightarrow \phi(n) = T(n)$.

That is n is perfect iff number of generators of the group $(Z_n, +)$ equals to $T(n)$.

In the previous example it was, $T(28) = 12 = \phi(28)$, thus the number $n = 28$ is a perfect number.

For the importance of the value of $T(n)$, we provide some formulas for $T(n)$.

Lemma: 3 If p prime then, $T(p) = 0,\ and\ T(p^k) = p^{k-2} + p^{k-3} + \cdots + p + 1,\ for\ k \geq 2.$

Proof: $T(p) = \varphi(p) - G(p)$
$= (p-1) - (p-1) = 0.$

For $k \geq 2$, we have

$$T(p^k) = (p^k - p^{k-1}) - (p^k - p^{k-1} - p^{k-2} - \cdots - p - 1)$$
$$= p^{k-2} + p^{k-3} + \cdots + p + 1.$$

In particular $T(p^2) = 1$.

Theorem: 4. If $(n, m) = 1,\ then\ T(nm) = T(n)\varphi(m) + T(m)\varphi(n) + 2\ S(n)S(m) - T(n)T(m).$

Proof: $G(nm) = G(n)G(m) - 2\ S(n)S(m)$

$$= (\varphi(n) - T(n))(\varphi(m) - T(m)) - 2\ S(n)S(m)$$
$$= \varphi(nm) - [\ \varphi(n)T(m) + \varphi(m)T(n) + 2\ S(n)S(m) - T(n)T(m)\ ].$$

and since $G(nm) = \phi(nm) - T(nm)$, the result follows.

In particular if p doesn't divide m, then $T(pm) = (p-1)T(m) + 2\ S(m).$

Theorem: 5 If p doesn’t divide m, then
$T(p^k m) = p.T(p^{k-1}m) + \sigma(m)\ \ for\ k \geq 2.\ And\ \ T(pm) = (p-1)T(m) + 2\ S(m).$

Proof: If $k \geq 2,$ by Corollary 1, we have

$$G(p^k m) = p.G(p^{k-1}m) - \sigma(m)$$
$$= p\ [\phi(p^{k-1}m) - T(p^{k-1}m)] - \sigma(m)$$
$$= \phi(p^k\ m) - p.T(p^{k-1}m) - \sigma(m).$$

And since $G(p^k m) = \phi(p^k m) - T(p^k m),$ the result follows.

If $k = 1,$ the result follows directly by Theorem 4.

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