International Research Journal of Pure Algebra -4(3), 2014, 440-443
Available online through www.rjpa.info ISSN 2248–9037

FIXED POINT THEOREMS

Naseer Ahmad Gilani*

P. G. Department of Mathematics, Govt. College Baramulla, Kashmir (India).

(Received on: 30-01-14; Revised & Accepted on: 27-03-14)

ABSTRACT

There is a conjecture of S. Reich which concerns with the existence of fixed points of multivalued mappings that satisfy a certain contractive condition. N. Mizoguchi and W. Takahashi has provided a positive answer to this conjecture of S. Reich. In this paper, we will give an alternative proof for the theorem of N. Mizoguchi and W. Takahashi.

AMS Subject Classification: *47H10, 54H25.*

Keywords: *Proximinal Sets, Compactly Positive Mappings, Weakly Contractive Mappings, Complete Metric Space, Hausdorff Metric Space and Fixed Point.*

1. INTRODUCTION AND STATEMENT OF RESULTS

Suppose (X, d) be a metric space. Now we have the following definitions.

Definition: 1 A metric space (X, d) is said to be complete if every Cauchy sequence converges to a point in X.

Definition: 2 Let P be a subset of X. Then P is said to be proximinal if for each x ϵX, ∃ an element pϵP such that

$$d(x, p) = d(x, P)$$

where
d(x, P) = inf.{d(x, y): yϵ P}.

The family of all bounded proximinal subsets of X is denoted by Q(X). Now we represent the family of all non- empty closed and bounded subsets of X by CB(X).

Definition: 3 A mapping f: X x X $\rightarrow [0,\infty)$ is said to be compactly positive if
Inf. {f(x, y): a $\leq d(x,y) \leq b\} > 0$ for each finite interval $[a,b]$ contained in $(0,\infty)$.

Definition: 4 A mapping T: X→CB(X) is said to be weakly contractive if ∃ a compactly positive mapping f such that

H{T(x), T(y)}$\leq d(x,y) - f(x,y)$ for each x, y ϵ X, where H is a Hausdorff metric on CB(X) induced by d.

Definition: 5 A fixed point of a function f from a set S to itself is a point x in S such that

$$f(x) = x$$

Now we mention the following lemma of [2].

Lemma: 1.1 Let T: X→Q(X) be a mapping then the following statements are equivalent;

(a) T is weakly contractive.
(b) H{T(x), T(y)} ≤ h(x, y) d(x, y) for some non-negative function h that satisfies
Sup {h(x, y): a $\leq d(x,y) \leq b\} < 1$ for each finite closed interval $[a,b]$ contained in $(0,\infty)$.
(c) H{T(x), T(y)}≤ φ(x,y), where φ is such that d – φ is compactly positive.

Dugundji and Granas [1] proved that a single-valued weakly contractive mapping of a complete metric space into itself has a unique fixed point.

Corresponding author: Naseer Ahmad Gilani
P. G. Department of Mathematics, Govt. College Baramulla, Kashmir (India).
E-mail: geelanina@gmail.com

By using (b) from lemma1.1 above for the weakly contractive mapping, Kaneko [2] provided a partial generalization (Theorem 1.3) of the theorem of Dugundji and Granas to the multivalued mappings. But till date, a complete generalization is not available in the literature. In [2], the below mentioned two theorems were proved.

Theorem: 1.2 Suppose (X, d) be a complete metric space and T: ~~X~~ Q(X) be a mapping. Let λ be a monotonic increasing function with $0 \leq \lambda(t) < 1$ for each $t \epsilon (0,\infty)$ and if $H\{T(x), T(y)\} \leq \lambda(d(x,y))\, d(x,y)$ for each x, y ϵ X, then T has a fixed point in X.

Theorem: 1.3 Suppose (X, d) be a complete metric space and T: X⟶ Q(X) be such that

$H\{T(x), T(y)\} \leq h(x,y)\, d(x, y)$ for each x, y ϵ X and for some non-negative function h that satisfies
$\text{Sup.}\{h(x, y): a \leq d(x,y) \leq b\} < 1$ for each finite closed interval$[a,b]$ contained in $(0,\infty)$. We also assume that if (x_n, y_n) ϵ X x X is such that

$\lim_{n\to\infty} d\,(x_n, y_n) = 0$, then $\lim_{n\to\infty} h\,(x_n, y_n) = k$, for some k ϵ [0,1), then T has a fixed point in X.

The above two theorems i.e., 1.2 and 1.3 were investigated in response to a problem which was put forth by S. Reich. Reich [4] proposed the following problem.

Conjecture: 1.4 Suppose (X, d) be a complete metric space and T: ~~X~~⟶ CB(X) satisfies the condition $H\{T(x), T(y)\} \leq k(\, d(x, y)\,)\, d(x, y)$ for all x, y ϵ X, $x \neq y$ where k: $(0,\infty) \to [0,1]$ and $\lim_{r\to t^+} \sup k(r) < 1$ for all $0 < t < \infty$. Then T a fixed point in X.

The above conjecture has been proven valid in an almost complete form by Mizoguchi and Takahashi [3]. But both of them replaced the condition on k by a stronger condition given below;

$$\lim_{r\to t^+} \sup k(r) < 1 \text{ for all } 0 \leq t < \infty.$$

But in this paper we reaffirm this positive response by Mizoguchi and Takahashi to the conjecture of Reich by giving an alternative proof.

2. MAIN RESULTS

Our main purpose in this paper is to prove Theorem 2.1onfixed points.

Theorem: 2.1 Suppose (X, d) be a complete metric space and T: X ⟶ CB(X) be a mapping. If λ is a function of $(0,\infty)$ to $[0,1)$ such that $\lim_{r\to t^+} \sup \lambda(r) < 1$ for all $0 \leq t < \infty$ and if $H\{T(x), T(y)\} \leq \lambda(\, d(x, y)\,)\, d(x, y)$ for all x, y ϵ X, then T has a fixed point in X.

Proof of Theorem: 2.1

We consider two points x_0 and x_1 such that x_0 ϵ X and $x_1 \epsilon T(x_0)$. Let us take a positive integer n_1 such that

$$\lambda^{n_1}\{d(x_0, x_1)\} \leq [1 - \{d(x_0, x_1)\}]\, d(x_0, x_1)$$

Again if we select x_2 in $T(x_1)$, using the definition of Hausdorff metric so that

$d(x_2, x_1) \leq H\{T(x_1), T(x_0)\} + \lambda^{n_1}\{d(x_0, x_1)\}$, then we have

$d(x_2, x_1) \leq \lambda\{d(x_1, x_0)\}\, d(x_1, x_0) + \lambda^{n_1}\{d(x_0, x_1)\} < d(x_1, x_0)$.

Now let us choose a positive integer n_2 s. t. $n_2 > n_1$ so that

$\lambda^{n_2}\{d(x_2, x_1)\} < [1 - \{d(x_2, x_1)\}]\, d(x_2, x_1)$

Because $T(x_2)$ ϵ CB(X), we select $x_3 \epsilon T(x_2)$ so that

$d(x_3, x_2) \leq H\{T(x_2), T(x_1)\} + \lambda^{n_2}\{d(x_2, x_1)\}$, then we have

$$\begin{aligned} d(x_3, x_2) &\leq H\{T(x_2), T(x_1)\} + \lambda^{n_2}\{\, d(x_2, x_1)\} \\ &\leq \lambda\,\{d(x_2, x_1)\}\, d(x_2, x_1) + \lambda^{n_2}\{\, d(x_2, x_1)\} \\ &< d(x_2, x_1) \end{aligned}$$

We repeat this process, since $T(x_k) \in CB(X)$ for each k, we may select a positive integer n_k such that

$$\lambda^{n_k}\{d(x_k, x_{k-1})\} \leq [1 - \{d(x_k, x_{k-1})\}]\, d(x_k, x_{k-1}).$$

Let us select x_{k+1} in $T(x_k)$ so that

$d(x_{k+1}, x_k) \leq H\{T(x_k), T(x_{k-1})\} + \lambda^{n_k}\{d(x_k, x_{k-1})\}$ then $d(x_{k+1}, x_k) < d(x_k, x_{k-1})$ so that $d_k \equiv d(x_k, x_{k-1})$ is a monotone non-increasing sequence of non-negative numbers.

Now let us prove that the sequence $\langle d_k \rangle$ is a Cauchy sequence.

Suppose $\lim_{k\to\infty} d_k = u \geq 0$. But by assumption, $\lim_{t\to u^+} \sup \lambda(t) < 1$. Therefore, $\exists$ a point k_0 such that $k \geq k_0$ implies that $\lambda(d_k) < h$, where $\lim_{t\to u^+} \sup \lambda(t) < h < 1$.

Now we take d_{k+1} .

$$d_{k+1} = d(x_{k+1}, x_k) \leq H\{T(x_k), T(x_{k+1})\} + \lambda^{n_k}(d_k)$$

$$\leq \lambda(d_k)\, d_k + \lambda^{n_k}(d_k)$$

$$\leq \lambda(d_k)\, \lambda(d_{k-1})\, d_{k-1} + \lambda(d_k)\, \lambda^{n_{k-1}}(d_{k-1}) + \lambda^{n_k}(d_k)$$

……………………………………………….and so on.

$$\leq \prod_{i=1}^{k} \lambda(d_i)\, d_1 + \sum_{m=1}^{k-1} \prod_{i=m+1}^{k} \lambda(d_i)\, \lambda^{n_m}(d_m) + \lambda^{n_k}(d_k)$$

$$\leq \prod_{i=1}^{k} \lambda(d_i)\, d_1 + \sum_{m=1}^{k-1} \prod_{i=max\,(k_0, m+1)}^{k} \lambda(d_i)\, \lambda^{n_m}(d_m) + \lambda^{n_k}(d_k) = R \text{ (say)}.$$

In the last inequality, we deleted some λ factors from the product because of the fact that $\lambda < 1$.

Now we take

$$\sum_{m=1}^{k-1} \prod_{i=max\,(k_0, m+1)}^{k} \lambda(d_i)\, \lambda^{n_m}(d_m) \leq (k_0 - 1)\, h^{k-k_0+1} \sum_{m=1}^{k_0-1} \lambda^{n_m}(d_m) + \sum_{m=k_0}^{k-1} h^{k-m}\, \lambda^{n_m}(d_m)$$

$$\leq (k_0 - 1)\, h^{k-k_0+1} \sum_{m=1}^{k_0-1} \lambda^{n_m}(d_m) + \sum_{m=k_0}^{k-1} h^{k-m+n_m}$$

$$\leq G\, h^k + \sum_{m=k_0}^{k-1} h^{k-m_{n_m}}$$

$$\leq G\, h^k + h^{k+n_{k_0}-k_0} + h^{k+n_{k_0-1}-(k_o-1)} + \ldots\ldots + h^{k+n_{k-1}-(k-1)}$$

$$\leq G\, h^k + \sum_{m=k+n_{k_0}-k_0}^{k+n_{k-1}-(k-1)} h^m$$

$$= G\, h^k + \frac{h^{k+n_{k_0}-k_0+1} - h^{k+n_{k-1}-k+2}}{(1-h)}$$

$$= G\, h^k + h^k \left\{\frac{h^{n_{k_0}-k_0+1} - h^{n_{k-1}-k+2}}{(1-h)}\right\}$$

$$= G\, h^k + h^k \left\{\frac{h^{n_{k_0}-k_0+1}}{(1-h)}\right\} = G\, h^k,$$

Where G is a generic positive constant.

Now we have

$$R \leq \prod_{i=1}^{k} \lambda(d_i)\, d_1 + G\, h^k + \lambda^{n_k}(d_k)$$

$$< h^{k-k_0+1} \prod_{i=1}^{k_0-1} \lambda(d_i)\, d_1 + G\, h^k + h^{n_k}$$

$< G\, h^k + G\, h^k + h^k = G_1\, h^k$, where G_1 is again a generic constant.

Now it is very easy to prove that the sequence $\langle x_k \rangle$ is a Cauchy sequence.

For $k \geq k_0$, $m \in N$,

$d(x_k, x_{k+m}) \leq d(x_k, x_{k-1}) + \ldots\ldots + d(x_{k+m-1}, x_{k+m})$

$$= \sum_{i=k+1}^{k+m} d_i < \sum_{i=k+1}^{k+m} G\, h^{i-1} = G\{\frac{h^{k+1} \ - h^{k+m}}{(1-h)}\} \leq h^k \to 0 \text{ as } k \to \infty.$$

Suppose $x_k \to x \in X$, then

$d\{ x, T(x) \} \leq d(x, x_k) + d(x_k, T(x))$

$\leq d(x, x_k) + H\{T(x_{k-1}), T(x)\}$

$\leq d(x, x_k) + \lambda \{ d(x_{k-1}, x)\}\, d(x_{k-1}, x)$

Since both the terms in the above expression tend to zero as $k \to \infty$, we get T(x) = x.

This shows that T has a fixed point in X, which proves the desired theorem 2.1 completely.

Corollary: Suppose (X, d) be a complete metric space and T: X → CB(X) be a mapping.

Let be a monotonic increasing function such that 0≤ (t) < 1 for each t ϵ (0, ∞) and if

$H\{T(x), T(y)\} \leq \lambda(d(x, y))\, d(x, y)$ for all x, y ϵ X, then T has a fixed point x in X.

The above corollary generalizes theorem 1.2 by extending the range of T from Q(X) to CB(X).

REFERENCES

[1] J. Dugundji and A. Granas, weakly contractive maps and elementary domain invariance theorem, *Bull. Greeks Math. Soc. 19 (1978), 141-151*.

[2] H. Kaneko, Generalized contractive multi-valued mappings and their fixed points, *Math. Japon. 33 (1988), 57-64.*

[3] N. Mizoguchi and W. Takahashi, fixed point theorems for multi-valued mappings on complete metric spaces, *J. Math. Anal. Appl. 141 (1989), 177-188.*

[4] S. Reich, Some problems and results in fixed point theory, *Contemp. Math. 21 (1983), 179-187.*

Source of Support: Nil, Conflict of interest: None Declared