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NONBONDAGE AND TOTAL NONBONDAGE NUMBERS IN DIGRAPHS

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ABSTRACT

Let $D= (V, A)$ be a digraph. A set S of vertices in a digraph D is called a dominating set of D if every vertex v in V – S, there exists a vertex u in S such that (u, v) in A. The domination number $\gamma(D)$ of D is the minimum cardinality of a dominating set of D. A set S of vertices in a digraph D is called a total dominating set of D if S is a dominating set of D and the induced subdigraph $\langle S\rangle$ has no isolated vertices. The total domination number $\gamma_t(D)$ of D is minimum cardinality of a total dominating set of D. The nonbondage number $b_n(D)$ of a digraph D is the maximum cardinality among all sets of arcs $X \subseteq A$ such that $\gamma(D - X) = \gamma(D)$. The total nonbondage number $b_{tn}(D)$ of a digraph D without isolated vertices is the maximum cardinality among all sets of arcs $X\subseteq A$ such that $D - X$ has no isolated vertices and $\gamma_t(D - X) = \gamma_t(D)$. In this paper, the exact value of $b_n(D)$ for any digraph D is found. We obtain several bounds on the bondage and total nonbondage numbers of a graph. Also exact values of these two parameters for some standard graphs are found.

Keywords: *digraph, nonbondage number, total nonbondage number.*

Mathematics Subject Classification: *05C.*

1. INTRODUCTION

In this paper, $D = (V, A)$ is a finite directed graph without loops and multiple arcs (but pairs of opposite arcs are allowed) and $G=(V, E)$ is a finite, undirected graph without loops multiple edges. For basic terminology, we refer to Chartand and Lesnaik [3].

A set S of vertices in a graph G is a *dominating set* if every vertex in $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A recent survey of $\gamma(G)$ can be found in Kulli [8].

Among the various applications of the theory of domination that have been considered, the one that is perhaps most often discussed concerns a communication network. Such a network consists of existing communication links between a fixed set of sites. The problem is to select smallest set of sites at which to place transmitters so that every site in the network that does not have a transmitter is joined by a direct communication link to one that does have a transmitter. To minimize the direct communication links in the network, in [17] Kulli and Janakiram introduced the concept of the nonbondage number in graphs as follows:

The nonbondage number $b_n(G)$ of a graph G is the maximum cardinality among all sets of edges $X\subseteq E$ such that $\gamma(G - X) = \gamma(G)$.

This concept was also studied in [9, 10, 11, 12, 13, 14, 15, 16].

Let G be a graph without isolated vertices. A dominating set S of V is called a *total dominating set* of G if the induced subgraph $\langle S\rangle$ has no isolated vertices. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G.

In [9], Kulli introduced the concept of total nonbondage in graphs as follows:

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The *total nonbondage number* $\gamma_{tn}(G)$ of a graph G without isolated vertices is the maximum cardinality among all sets of edges $X \subseteq E$ such that $G - X$ has no isolated vertices and $\gamma_t(G - X) = \gamma_t(G)$.

Let $D = (V, A)$ be a digraph. For any vertex $u \in V$, the sets $O(u)=\{v/(u, v) \in A\}$ and $I(u)=\{v/(v, u) \in A\}$ are called the outset and inset of u. The indegree and outdegree of u are defined by $id(u) = |I(u)|$ and $od(u) = |O(u)|$. The maximum outdegree of D is denoted by $\Delta^+(D)$. Let $\lceil x \rceil (\lfloor x \rfloor)$ denote the least (greatest) integer greater (less) than or equal to x.

A set S of vertices in a digraph $D=(V, A)$ is a *dominating set* if for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that $(v, u) \in A$. The *domination number* $\gamma(D)$ of D is the minimum cardinality of a dominating set of D.

Let $D = (V, A)$ be a digraph in which $id(v)+od(v) > 0$ for all $v \in V$. A subset S of V is called a *total dominating set* of D if S is a dominating set of D and the induced subdigraph $\langle S \rangle$ has no isolated vertices. The *total domination number* $\gamma_t(D)$ of D is the minimum cardinality of a total dominating set of D, (see [1]).

In [2, 6], the concept of bondage in digraphs was studied and in [7], the concept of total bondage in digraphs was studied.

In this paper, we introduce the analog of nonbondage and total nonbondage in digraphs. We obtain several results on these parameters.

2. NONBONDAGE NUMBER IN DIGRAPHS

The concept nonbondage number can be extended to digraphs.

Definition: 2.1 The *nonbondage number* $b_n(D)$ of a digraph $D=(V, A)$ is the maximum cardinality among all subsets of arcs $X \subseteq A$ such that $\gamma(D - X) = \gamma(D)$.

Since the domination number of every spanning subgraph of a digraph D is at least $\gamma(D)$, the nonbondage number of a nonempty digraph is well defined.

A γ-set is a minimum dominating set and a b_n-set is a maximum nonbondage set of D.

Remark: 2.2 *In the definition 2.1, if* $X=\phi$, *then* $b_n(D)=0$.

Proposition: 2.3 *Let* $K_{1,p}$ *be a directed star in which* $od(u)=p$ *and* $id(u_i)=1$, $1 \le i \le p$. *Then*

$$b_n(K_{1,p})=0.$$

Proof: Clearly $\gamma(K_{1,p}) = 1$. Also $\gamma(K_{1,p} - uu_i) = 2$ for $1 \le i \le p$. Thus $b_n(K_{1,p})=0$.

Proposition: 2.4 *Let* $K_{1,p}$ *be a directed star in which* $od(u_i)=1$, $1 \le i \le p$ *and* $id(u)=p$. *Then*

$$b_n(K_{1,p})=p - 1.$$

Proof: Clearly $\gamma(K_{1,p})=p$. Let $uu_i=e_i$ be arcs of $K_{1,p}$. Then

$$\gamma(K_{1,p} - \{e_1, \dots , e_{p-1}\}) = p.$$

and $$\gamma(K_{1,p} - \{e_1, \dots , e_p\}) = p+1.$$

Thus $$b_n(K_{1,p})=p - 1.$$

Proposition: 2.5 *For a directed path* P_p *with* $p \ge 3$ *vertices,*

$$b_n(P_p) = \frac{p}{2} - 1, \quad \text{if } p \text{ is even,}$$
$$= \left\lfloor \frac{p}{2} \right\rfloor, \quad \text{if } p \text{ is odd.}$$

Proof: Let $P_p = (v_1, v_2, \dots , v_p)$ be a directed path with $p \ge 3$ vertices. Let $v_iv_{i+1}=e_i$ be directed arcs, $1 \le i \le p - 1$.

We consider the following two cases.

Case: 1 Suppose p is even. Then the removal of set of arcs $X_1=\{e_2, e_4, ..., e_{p-2}\}$ from P_p results in a digraph D_1 containing only $\frac{p}{2}$ isolated arcs. Thus

$$\gamma\left(P_p - X_1\right) = \gamma\left(D_1\right) = \gamma\left(P_p\right) = \frac{p}{2}.$$

Also $\left|X_1\right| = \frac{p-2}{2}$. Furthermore, the removal of any arc e from D_1 results a digraph such that $\gamma\left(D_1 - e\right) = \frac{p}{2}$. Thus $b_n\left(P_p\right) = \left|X_1\right| = \frac{p}{2} - 1.$

Case: 2 Suppose p is odd. Then the removal of set of arcs $X_2=\{e_2, e_4, ..., e_{p-1}\}$ from P_p results in a digraph D_2 containing only $\frac{p-1}{2}$ isolated arcs and an isolated vertex. Thus

$$\gamma\left(P_p - X_2\right) = \gamma\left(D_2\right) = \gamma\left(P_p\right) = \frac{p-1}{2} + 1 = \left\lceil \frac{p}{2} \right\rceil.$$

Also, $\left|X_2\right| = \frac{p-1}{2}$. Furthermore, the removal of any arc e from D_2 results a digraph such that $\gamma\left(D_2 - e\right) > \left\lceil \frac{p}{2} \right\rceil$.

Thus $b_n\left(P_p\right) = \left|X_2\right| = \frac{p-1}{2} = \left\lfloor \frac{p}{2} \right\rfloor$.

Proposition: 2.6 *For any directed cycle C_p with $p\geq 3$ vertices,*

$$b_n\left(C_p\right) = \frac{p}{2}, \qquad \text{if } p \text{ is even,}$$
$$= \left\lfloor \frac{p}{2} \right\rfloor + 1, \text{ if } p \text{ is odd.}$$

Proof: Let C_p be a directed cycle with $p\geq 3$ vertices. Since $C_p - e = P_p$ for any arc e of C_p, we have

$$\gamma\left(C_p\right) = \gamma\left(C_p - e\right) = \frac{p}{2}, \qquad \text{if } p \text{ is even,}$$
$$= \left\lfloor \frac{p}{2} \right\rfloor + 1, \qquad \text{if } p \text{ is odd.}$$

Thus $b_n(C_p)>1$ and $b_n(C_p) = 1 + b_n(P_p)$. Thus by proposition 2.5,

$$b_n\left(C_p\right) = \frac{p}{2}, \qquad \text{if } p \text{ is even,}$$
$$= \left\lfloor \frac{p}{2} \right\rfloor + 1, \quad \text{if } p \text{ is odd.}$$

Theorem: 2.7 *For any digraph D with p vertices and q arcs,*

$$b_n(D) = q - p + \gamma(D). \qquad (1)$$

Proof: Let S be a γ-set of D. For each vertex v in $V - D$, choose exactly one arc (u, v) which is incident to v and to a vertex u in S. Let X be the set of all such arcs. Then clearly $A - X$ is a b_n-set of D. Thus (1) holds.

Theorem: A*[1] For any digraph D with p vertices,*

$$\gamma(D) \leq p - \Delta^+(D). \qquad (2)$$

We obtain an upper bound for $b_n(D)$.

Theorem: 2.8 *For any digraph D with p vertices and q arcs,*

$$b_n(D) \leq p - \Delta^+(D). \tag{3}$$

Proof: This follows from (1) and (2).

Theorem: 2.9 *For any subdigraph H of a digraph D,*

$$b_n(H) \leq b_n(D). \tag{4}$$

Proof: Since every nonbondage set of H is a nonbondage set of D, (4) holds.

Corollary: 2.10 *For any digraph D with p vertices, which has a hamiltonian circuit,*

$$b_n(D) \geq \left\lceil \frac{p}{2} \right\rceil. \tag{5}$$

Proof: This follows from (4) and the fact C_p is a spanning subdigraph of D and $\gamma(C_p) = \left\lceil \frac{p}{2} \right\rceil$.

The following result gives a new upper bound for $b(D)$.

Theorem: 2.11 *For any digraph D,*

$$b(D) \leq b_n(D)+1 \tag{6}$$

and this bound is sharp.

Proof: Let X be a b_n-set of a digraph D. Then, for any arc e in $D - X$, $X \cup \{e\}$ is a bondage set of D. Thus

$$b(D) \leq |X \cup \{e\}|.$$

This prove (6).

The equality in (6) holds if $K_{1,p}$ is a directed star in which $od(u_i)=1$, $1 \leq i \leq p$, and $id(u)=p$.

The following result is another upper bound for $b(D)$.

Corollary: 2.12 *For any digraph D,*

$$b(D) \leq q - \Delta^+(D)+1. \tag{7}$$

Proof: This follows from (6) and (3).

3. TOTAL NONBONDAGE NUMBER IN DIGRAPHS

The concept of total nonbondage number can be extended to digraphs.

Definition: 3.1 The *total nonbondage number* $b_{tn}(D)$ of a digraph D without isolated vertices is the maximum cardinality among all subsets of arcs $X \subseteq A$ such $D - X$ has no isolated vertices and $\gamma_t(D - X) = \gamma_t(D)$.

A b_{tn}-set is a maximum total nonbondage set of D.

Remark: 3.2 *In the definition 3.1, if* $X=\phi$, *then* $b_{tn}(D)=0$.

Proposition: 3.3 *If* $K_{1,p}$ *is a directed star, then* $b_{tn}(K_{1,p})=0$.

Proof: Let $K_{1,p}$ be a directed star. Then for every arc a in $K_{1,p}$, $K_{1,p} - a$ has an isolated vertex. Thus $b_{tn}(K_{1,p}) = 0$.

Proposition: 3.4 *For a directed path* P_p *with* $p \geq 2$ *vertices,*

$$b_{tn}(P_p) = \begin{cases} 0, & \text{if } p = 2, 3, 4, \\ \left\lceil \frac{p}{2} \right\rceil - 2, & \text{if } p \geq 5. \end{cases}$$

Proposition: 3.5 *For a directed cycle* C_p *with* $p \geq 3$ *vertices,*

$$b_{tn}(C_p) = \left\lfloor \frac{p}{3} \right\rfloor.$$

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