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On Semi π-regular clean Ring

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ABSTRACT

In this paper we introduce the notion of semi π -regular clean rings. Some properties of semi π-regular clean ring are investigated, which generalize the well-known results of clean ring, and it's connection with other rings are given.

Keywords: *clean ring, almost clean ring, semi π –regular ring.*

1. INTRODUCTION

Throughout this paper R denotes an associative ring with identity. We use the symbol $r(a)$ to denote the right annihilator of a in R.

Following Han and Nicholson [5], an element x of a ring R is called clean if x can be written as the sum of a unit and an idempotent. A ring R is said to be clean if every element of R is clean. The concept of clean ring was first introduced by Nicholson [6] as early as 1977. Since then some stronger concepts (e.g. strongly clean, uniquely clean and weakly clean) have been considered, see [1, 2, 8]. In this work we consider a ring with every element is the sum of semi π-regular element and an idempotent element. We call such ring semi π-regular clean ring.

2. BASIC PROPERTIES

We start this section with the following definitions.

Definition: 2.1 A ring R is said to be a right semi π-regular ring if for all a in R, there exist a positive integer n and b in R such that $a^n = a^n b$ and $r(a^n) = r(b)$, [7].

Clearly b is idempotent, since $a^n = a^n b$, implies $a^n(1-b) = 0$, then $1-b \in r(a^n) = r(b)$.

Definition: 2.2 A ring R is said to be a right semi π-regular clean ring if every element of R can be written as the sum of a right semi π-regular element and idempotent element.

Next, we shall give the following result.

Lemma: 2.3 If a is a semi π-regular element, then$-a$ is also a semi π-regular element.

Proof: Let a be a semi π-regular element in R. Then there exist a positive integer n and $b$in R, such that $a^n = a^n b$ and $r(a^n) = r(b)$.

Now $-a^n = -a^n b$, then clearly $r(-a^n) = r(b)$

Proposition: 2.4 An element x in a ring R is aright semi π-regular clean iff $1-x$ is a right semi π-regular clean element.

Proof: Since x is a right semi π-regular clean element, then $x = e + a$, where e is idempotent and a is a right semi π-regular clean element.

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Hence there exist a positive integer n and $b$in R, such that $a^n = a^n b$ and $r(a^n) = r(b)$.

Now,

$$1 - x = 1 - (e + a)$$
$$= (1 - e) + (-a).$$

Since $1 - e$ is idempotent and $-a$ is a right semi π- regular element by (lemma.2.3.).Then $1 - x$ is a right semi π-regular clean element.

Conversely, assume that $1 - x$ is a right semi π-regular clean element, then $1 - x = e + a$, where e is idempotent and a is a right semi π- regular element. Then

$$x = 1 - e + (-a)$$

Thus x is a right semi π- regular clean element.

We next turn to give the following main result which characterize semi π-regular clean ring in terms of the right annihilator of an element in R.

Theorem: 2.5 A ring R is a right semi π-regular clean ring if and only if $r(a^n)$ is a direct summand for every a in R and some positive integer n.

Proof: Let R be a right semi π-regular clean ring and let x in R, then $x = e + a$, where e is idempotent element and a is a right semi π-regular element. Then there exist a positive integer n, and b in R such that $a^n = a^n b$ and $r(a^n) = r(b)$. Now $a^n = a^n b$ gives $a^n(1 - b) = 0$, and this implies $1 - b \in r(a^n)$. If we set $1 = 1 - b + b$, then $R = r(a^n) + bR$. Next we shall prove that $(a^n) \cap bR = (0)$. Let $y \in r(a^n) \cap bR$, then $y = br$ and $a^n y = 0$, for some r in R. This implies that $a^n br = 0$, and hence $a^n r = 0$, yields $r \in r(a^n) = r(b)$. So $br = 0$, and hence $y = 0$. Therefore $R = r(a^n) + bR$.

Conversely, assume that $r(a^n)$ is a direct summand, then there exists a right ideal I of R, such that $r(a^n) + I = R$. In particular, there exist $b \in r(a^n)$ and $i \in I$ such that $b + i = 1$. Multiply from the left by a^n, we get $a^n i = a^n$. We claim that $r(a^n) = r(i)$. Let $x \in r(a^n)$, then $a^n x = 0$, and hence $a^n ix = 0$, this implies $ix \in r(a^n)$, but $ix \in I$, thus $ix \in r(a^n) \cap I = (0)$, therefore $ix = 0$, so $x \in r(i)$.

Now let $y \in r(i)$, then $iy = 0$, and hence $a^n iy = 0$, so $a^n y = 0$ gives $y \in r(a^n)$. Whence it follows that $r(a^n) = r(i)$. On the other hand every element of R can be written as the sum of 0 and semi π- regular element. Therefore R is a right semi π-regular clean ring.

3. CONNECTION BETWEEN SEMI π-REGULAR CLEAN RINGS AND OTHER RINGS

In this section we explore the relation between a right semi π-regular clean ring with clean rings and almost clean rings.

Following [4], a ring R is said to be almost clean ring, if every element of R is the sum of a non-zero divisor element and an idempotent element.

We next turn to prove the following main result.

Theorem: 3.1 let R be a right semi π-regular ring with central idempotent, then R is almost clean ring.

Proof: Let a be a non-zero element in R, then there exist a positive integer n, and b in R, such that $a^n = a^n b$ and $r(a^n) = r(b)$.

If we set $c = (b - 1) + a$, we shall prove that c is a non-zero divisor.

Suppose that that $cy = 0$. Then $(b - 1 + a)y = 0$, this implies

$(b - 1)y = -ay$, since b is idempotent, then $-bay = 0$, so $aby = 0$

($b$is central). This gives $a^n by = 0$, and hence $a^n y = 0$.

Thus, $y \in r(a^n) = r(b)$, gives $by = 0$. Now $(b - 1 + a)y = 0$, implies $y = ay$. But $a^n y = 0$ gives $a^{n-1}y = a^n y = 0$. Repeat this process $n - 1$ times we get $y = 0$. Thus c is a non-zero devisor, whence it follows that $a = (1 - b) + c$

where $1-b$ is idempotent and c is a non-zero divisor.

Therefore R is an almost clean ring.

The following result is an immediate consequence of Theorem 3.1.

Corollary: 3.2 Let R be a right semi π-regular clean ring with central idempotent and every pair of idempotent are orthogonal, then R is almost clean ring.

Proof: Let R be aright semi π-regular clean ring and let $x \in R$, then $x = e + a$ where e is idempotent and a is a right semi π-regular element. By Theorem 3.1.$a = e_1 + c$, where e_1 is idempotent and c is anon-zero divisor then $x = e + e_1 + c$. Now, since $ee_1 = 0$ then $e + e_1$ is idempotent. Hence x is almost clean element.

Corollary: 3.3 Let R be a semi π-regular ring, then for every $a \in R$, there exist a positive integer n and a non-zero divisor c and idempotent b such that $a^n = bc$.

Proof: Let a be a non-zero divisor element of R, then there exist a positive integer n and b in R, such that $a^n = a^n b$ and $r(a^n) = r(b)$. If we set $c = b - 1 + a^n$, then clearly c is a non-zero divisor. Hence $a^n = bc$.

Following [3], a ring R is said to be a nil-clean if every element of R is the sum of nilpotent element and idempotent element.

We end this paper by proving that.

Theorem 3.4: Let R be a right semi π-regular clean ring with only idempotent 0 and 1. Then R is clean ring or nil ring or almost clean ring.

Proof: Let x be a semi π -regular clean element of R ,Then $x = e + a$ where e is idempotent and a is a right semi π-regular element If $a = 0$, then $x = e = (1-e) + (2e-1)$ clearly $(1-e)$ is idempotent and $(2e-1)$ is a unit element .Hence R is a clean ring. If $a \neq 0$ then there exist a positive integer n and b in R such that $a^n = a^n b$ and $r(a^n) = r(b)$. Since b is idempotent, then $b = 0$ or 1. If $b = 0$, then$a^n = 0$, a is a nilpotent element, there for R is a nil-clean ring. On the other hand if $b = 1$, then $(a^n) = r(b) = r(1) = 0$. So a^n is a non-zero divisor. Hence R is a almost clean ring.

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