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ORDER TOPOLOGY AND UNIFORMITY ON A-METRIC SPACE

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ABSTRACT

"Order convergence" of a sequence $\{x_n\}$ is introduced in an A-metric space (X, A, d) [14]. Order Topology and its properties are studied in this space; we obtained a base for some Uniformity on X.

Key words: *A-metric space, Order convergence of sequence, Order closed sets, Order Topology.*

1. INTRODUCTION

In this paper, we introduced the "Order convergence" of a sequence $\{x_n\}$ in an A-metric space (X, A, d)[14], and we proved that Order Topology in A-metric space satisfies T_1-separation axiom and also we obtained a necessary and sufficient condition for a subset V of X to be open in the Order Topology on an A-metric space (X, A, d), in terms of a convergent sequence in X. If Y is any arbitrary Topological space, (X, A, d) is an A-metric space and $\phi : X \to Y$ is a mapping, then ϕ is continuous, if and only if, $x_i \to x \Rightarrow \phi(x_i) \to \phi(x)$.

Further, given an A-metric space (X, A, d), we obtained a base for some Uniformity on X, in such a way that, this base induces the usual Uniformity on any usual metric space (X, d), when it is viewed as a A-metric space (X, A, d), where $A = \mathbb{R}$ (set of real numbers).

2. ORDER TOPOLOGY AND A UNIFORMITY ON AN A-METRIC SPACE

In this section, we introduce "Order convergence," "Order closed set," "Order Topology" and a Uniformity on an A-metric space (X, A, d).

Definition: 2.1 A Lattice ordered Autometrized Algebra $A = (A, +, \leq, *)$ [10] is called a Representable Autometrized Algebra, if and only if, A satisfies the following:

(i) $A = (A, +, \leq, *)$ is a semi regular Autometrized algebra

(ii) for every $a \in A$, all the mappings $x \mapsto a + x, x \mapsto a \vee x, x \mapsto a \wedge x$ and $x \mapsto a * x$ are contractions (A mapping $f : A \to A$ is a contraction w.r.to $*$, if and only if, $f(x) * f(y) \leq x * y$ for all x, y in A).

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Definition: 2.2 Let X be a non empty set, let $A=(A,+,\leq,*)$ be a Representable Autometrized Algebra, let $d: X\times X\to A$ be a mapping satisfying the following properties of a distance function

$(M_1): d(a,b)\geq 0,$ for all a,b in X, with equality occurring, if and only if, $a=b$ (non-negativity)

$(M_2): d(a,b)=d(b,a),$ for all a,b in X (Symmetry)

$(M_3): d(a,c)\leq d(a,b)+d(b,c),$ for all a,b,c in X (triangle in equality)

Then, (X,A,d) is said to be an A-metric space.

Definition: 2.3 Let (X,A,d) be an A-metric space. A sequence $\{x_n\}$ of elements of X is said to converge "In Order" to an element x in X, if and only if, $d(x_n,x)\to 0$ in A, and in case, we write $x_n\to x.$

Result 2.4: In any A-metric space (X,A,d), we have the following

(i) $x_i\to x, x_i\to y \Rightarrow x=y$

(ii) $x_i\to x,$ then $x_{n_i}\to x$ for every subsequence $\{x_{n_i}\}$ of $\{x_i\}$.

Proof: Let (X,A,d) be an A-metric space.

(i) Let $\{x_n\}$ be a sequence in X, such that $x_i\to x$ and $x_i\to y$

Therefore $d(x_i,x)\to 0$ and $d(x_i,y)\to 0$.

But, by the triangle inequality of d, we have $d(x,y)\leq d(x,x_i)+d(x_i,y)$

Taking limit as $i\to\infty$, we have

$d(x,y)\leq 0+0$

$\Rightarrow d(x,y)\leq 0$

$\Rightarrow d(x,y)=0$, since $d(x,y)\geq 0$

$\Rightarrow x=y$

(ii) Proof: is obvious.

Let us introduce the following

Definition: 2.5 Let (X,A,d) be an A-metric space. A subset S of X is said to be "Order closed", if and only if, for every convergent sequence in S, the Order limit of the sequence is also a member of S.

Lemma: 2.6 Let (X,A,d) be any A-metric space. We have

(i) ϕ , X are Order closed in X

(ii) Arbitrary intersection of Order closed sets in X is also Order closed in X

(iii) Finite union of Order closed sets in X is also Order closed in X .

Proof: Let (X,A,d) be an A-metric space.

(i) Obviously, ϕ and X are Order closed subsets in X.

(ii) To each $i\in I$, let S_i be an Order closed subset of X.

Put $S=\bigcap_{i\in I} S_i$

Now, let $\{x_n\}$ be any convergent sequence in S and let $x_n \to x$.

$\Rightarrow x_n \in S,\ \forall\, n \in N$

$\Rightarrow x_n \in \bigcap_{i\in I} S_i, \forall\, n \in N$

$\Rightarrow x_n \in S_i\ \ \forall\, i \in I$ and $\forall\, n \in N$

But each S_i is an Order closed set of X

$\therefore\ x_n \in S_i\ \ \forall\, i \in I\ \Rightarrow x \in \bigcap_{i\in I} S_i = S$

$\therefore\ x \in S$

$\therefore\ S$ is also Order closed in X

(iii) Let $S_1, S_2, S_3, ..., S_n$ be Order closed sets in X

Put $S = \bigcup_{i=1}^{n} S_i$, Now, let $\{x_n\}$ be a sequence in S converging to x.

$\Rightarrow$ there exist some S_j which contains an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$.

But $x_n \to x \Rightarrow x_{n_j} \to x\ \ (\text{Result } 2.4\ (\text{ii}))$

But S_j is an Order closed set of X

$\therefore\ x \in S_j \Rightarrow x \in \bigcup_{i=1}^{n} S_i = S$

$\therefore\ S$ is an Order closed subset of X.

In view of the above Lemma2.6, it follows that the Order closed sets of an A-metric space (X, A, d) are exactly the closed sets of a certain Topology on X, the Order Topology.

Remark: 2.7 If we consider a metric space (X, d) as an A-metric (X, A, d) where $A = \mathbb{R}$, then, it is clear that the Order Topology coincides with the metric Topology.

Now, we show that the Order Topology on an A-metric space (X, A, d) satisfies the T_1- axiom and obtain a characterization of an open set in the Order Topology in terms of a convergent sequence.

Theorem: 2.8 Let (X, A, d) be an A-metric space with Order Topology in X.

(i) The Order Topology in X satisfies the T_1-separation axiom. i.e., every subset of X consisting of a single point, is Order closed.

(ii) A subset V of X is open in the Order Topology, if and only if, for every sequence $\{x_n\}_{n=1,2,3,....}$ in X converging to a point x in V, x_i is in V for all but a finite number of the x_i.

Proof: Let (X, A, d) be an A-metric space with Order Topology.

(i) let $\{x\}$ be any singleton subset of X, Put B= $\{x\}$ let $\{x_n\}$ be any convergent sequence in B converges to x_0 **(**say)

Since B is a singleton set, $x_n = x, \forall n$

$\therefore \{x_n\} = \{x, x, x, ...\}$ is a constant sequence.

Clearly, $x_n \to x, \Rightarrow x_0 = x = x_n \in$ B

$\therefore \{x\}$ is Order closed.

$\therefore$ Every subset of X consisting of a single point is Order closed in X.

Thus, the Order Topology in X satisfies $T_1 -$Separation Axiom.

(ii) Let $V \subseteq X$ and let V be open in the Order Topology in X.

Let $\{x_n\}$ *be a convergent sequence in* X *converges to the point* x in V

i.e., $x_n \to x$, put $C = X - V$, $\therefore$ C is Order closed in X.

But $x_n \in X \ \forall n \in N$ and $x \in V \Rightarrow x \notin C$

We have to prove that $x_i \in V$ for all but finite number of x_i.

Suppose, if possible, that this is not true.

$\Rightarrow$ there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Where $\{x_{n_i}\} \notin V$, *i.e.*, $\{x_{n_i}\}$ is a sequence in C.

Since $x_n \to x$, $\{x_{n_i}\}$ also tends to x, but C is Order closed .

$\therefore x \in C$ *i.e.*, $x \notin V$, this is a contradiction.

$\Rightarrow x_i \in V$ for all but a finite number of x_i.

Conversely, assume that $\phi \neq V \subseteq X$ such that for every sequence $\{x_n\}$ in X, converging to a point x in V, x_i is in V for all but finite number of x_i.

Now, we have to show that V is open in X.

$\therefore$ it is enough to prove that $C = X - V$ is Order closed in X.

So let $\{x_n\}$ be any convergent sequence in C, such that $x_n \to x$.

$\Rightarrow x_n \in C \ \forall n = 1, 2, 3, ...$

$\Rightarrow x_n \notin V$

Suppose, if possible that $x \notin C$ so that $x \in V$, thus, $\{x_n\}$ is converging sequence in X, converging to x in V,

$\Rightarrow x_i \in V$ for all but a finite number of x_i, which is a contradiction to the fact that $x_n \in C \ \forall n = 1, 2, 3, ...$ and $C \bigcap V \neq \phi$

$\therefore$ our supposition is false, hence , $x \in C$

$\Rightarrow C$ is Order closed in X.

$\Rightarrow V$ is Order open in X.

Hence the theorem.

Theorem 2.9: Let (X, A, d) be an A-metric space. Let Y be any arbitrary Topological space. A mapping $\phi: X \to Y$ is continuous, if and only if, $x_i \to x$ implies $\phi(x_i) \to \phi(x)$.

Proof: Let (X, A, d) be an A-metric space. Let (Y, τ) be any Topology space.

Let $\phi: X \to Y$ be a mapping.

First assume that $x_i \to x \Rightarrow \phi(x_i) \to \phi(x)$ (I)

We have to show that ϕ is continuous. So, let G be any open set in Y, now , let us show that $\phi^{-1}(G)$ is open in X.

For this, it is enough, if we prove that for every sequence $\{x_n\}$ in X converging to x in $\phi^{-1}(G)$, then $x_i \in \phi^{-1}(G)$ for all but a finite number of x_i.

So, let $\{x_n\}$ be a sequence in X converging to $x \in \phi^{-1}(G)$
$\therefore$ $\phi(x) \in G$, by **(I)**, $\phi(x_i) \to \phi(x) \in G., i.e., \{\phi(x_n)\}$ is a sequence in Y converging to $\phi(x) \in G$, which is open in Y.

$\Rightarrow \phi(x_i)$ is in G, for all but a finite number of the $\phi(x_i)$
$\Rightarrow x_i$ is in $\phi^{-1}(G)$ for all but a finite number of x_i.
$\Rightarrow \phi^{-1}(G)$ is open in X.

Thus, G is open in $Y \Rightarrow \phi^{-1}(G)$ is open in X.
$\therefore \phi$ is continuous.

Conversely, let us assume that $\phi: X \to Y$ is a continuous mapping,

Let $\{x_n\}$ be a convergent sequence in X, converging to x in X.
$\Rightarrow$ $\phi(x_n) \to \phi(x)$

Hence the Theorem.

Definition: 2.10 Let (X, A, d) be an A-metric space, let P be a proper dual ideal of the cone C of A satisfying the property (*): Given p in P, there exist q, r in P, such that $q + r \le p$, to each $p \in P$, write $U_p = \{(x, y) \in X \times X \ / \ d(x, y) < p\}$.

Theorem: 2.11 Let (X, A, d) be an A-metric space, let P be a proper dual ideal (lattice dual ideal) of the positive cone C of A satisfying the property (*) of the above definition **(2.9)**, then the family $\{U_p\}_{p \in P}$ forms a base for some Uniformity on X.

Proof: Let (X, A, d) be an A-metric space, let P be a proper dual ideal of the positive cone C of A.
$\tau = \{U_p \ / \ p \in P\}$ where $U_p = \{(x, y) \in X \times X \ / \ d(x, y) < p\}$ in order to show that τ is a base for some uniformity for X, it is enough if we prove the following:

(a) each element of τ contains the diagonal Δ, i.e., $\Delta \subseteq U_p, \forall\, p \in P$.

(b) $U_p^{-1} = U_p, \forall\, p \in P$

(c) $U_p \cap U_q \supset U_{p\wedge q}$

(d) Given $p \in P$, there exist q, r in P such that $U_p \circ U_r \subset U_p$

(a) We have $\Delta = \{(x,x) / x \in X\}$

Now, $(x,y) \in \Delta$

$\Rightarrow x = y$ and $x \in X$

$\Rightarrow d(x,y) = d(x,x) = 0$

$\Rightarrow d(x,y) < p\ \forall\ p \in P \qquad (\because P \subseteq C)$

$\Rightarrow (x,y) \in U_p\ \ \forall\ p \in P$

Thus, $\Delta \subseteq U_p, \forall\, p \in P$.

(b) Let $p \in P$,

$\therefore\ U_p = \{(x,y) \in X \times X / d(x,y) < p\}$

$\therefore U_p^{-1} = \{(y,x) / d(x,y) < p\}$

$= \{(x,y) / d(y,x) < p\} \quad (\because d \text{ is symmetry})$

$= U_p$

Thus, $U_p^{-1} = U_p, \forall\, p \in P$.

(c) Let $p \in P$ and $q \in P$

$\therefore\ U_{p\wedge q} = \{(x,y) / d(x,y) < p \wedge q\}$

Now, $(x,y) \in U_{p\wedge q}$

$\Rightarrow d(x,y) < p \wedge q$

$\Rightarrow d(x,y) < p$ and $d(x,y) < q \quad (\because\ p \wedge q < p\ \ \&\ p \wedge q < q\ )$

$\Rightarrow (x,y) \in U_p$ and $(x,y) \in U_q$

$\Rightarrow (x,y) \in U_p \cap U_q$

Thus, $U_p \cap U_q \supset U_{p\wedge q}$

(d) Let $p \in P$, by hypothesis P satisfies the Property **(*)**

$\therefore$ there exist q, r in P**,** such that $q + r \le p$.

Now, $(x,y) \in U_q \circ U_r$

$\Rightarrow$ there exist z in X such that $(x,z) \in U_r$ and $(z,y) \in U_q$

$\Rightarrow\ d(x,z) < r$ and $d(z,y) < q$

But by the triangle inequality, we have $d(x,y) \le d(x,z) + d(z,y)$

$\therefore d(x,y) < r + q < p \qquad (\because\ r + q = q + r < p)$

$\Rightarrow d(x, y) < p$

$\Rightarrow (x, y) \in U_p$

Thus, $U_p \circ U_r \subset U_{p.}$

Thus, $\tau = \{U_p / p \in P\}$ forms a base for some Uniformity on X.

Hence the Theorem.

Remark: 2.12 If (X, d) is any usual metric space and if we view it as an A-metric space (X, A, d). where A= $\mathbb{R}$ and if we write $P = \{x \in \mathbb{R} / x > 0\}$, then the family $\{U_p\}_{p \in P}$ is a base for the Usual Uniformity obtained on the given metric space (X, d).

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