International Research Journal of Pure Algebra -4(4), 2014, 495-500
Available online through www.rjpa.info ISSN 2248–9037

SOME SPECIAL NEAR – RINGS

G. Sugantha*[1] and R. Balakrishnan[2]

[1]Department of Mathematics, Pope's college, Sawyerpuram– 628251, India.

[2]P. G & Research Department of Mathematics, V. O. Chidambaram College, Thoothukudi-628 008, India.

(Received on: 10-04-14; Revised & Accepted on: 23-04-14)

ABSTRACT

In [4] ([5]) we defined a right near – ring N to be $\beta_1(\beta_2)$ if xNy = Nxy(xNy = xyN) for all x, y in N. Following these we make an attempt in this paper to study the properties of those near – rings which satisfy the conditions xNy = yxN and xNy = Nyx.

***Mathematics Subject Classification**: 16Y30.*

***Keywords**: β_3 and β_4 near – rings.*

1. INTRODUCTION

A right near – ring is an algebraic system (N,+,·) with two binary operations '+'and '·' such that
(i) (N, +) is a group with 0 as its identity.
(ii) (N, ·) is a semigroup and
(iii) (x + y) z = xz + yz for all x, y, z in N.

Throughout this paper, N stands for a right near – ring (N, +, ·) with at least two elements . Obviously, 0n = 0 for all n in N. As in [2], a subgroup (M, +) of (N, +) is called (i) a left N – subgroup of N if MN⊂M, (ii) an N – subgroup of N if NM⊂M and (iii) an invariant N – subgroup of N if M satisfies both (i) and (ii). Again in [2], N is defined to be weak commutative if xyz= xzy for all x, y, z in N. The concept of mate function in N has been introduced in [6] with a view to handling the regularity structure with considerable ease. A map '*f* 'from N into N is called (i) a mate function for N if *x* = *xf(x) x*. (ii) a P_3 mate function if in addition, *xf(x)* = *f(x)x* for all x in N. By identity 1of N, we mean only the multiplicative identity of N.

Basic concepts and terms used but left undefined in this paper can be found in [2].

2. NOTATIONS

(i) E denotes the set of all idempotents of N. (e in N is called an idempotent if $e^2 = e$)
(ii) L denotes the set of all nilpotents of N. (a in N is nilpotent if $a^k = 0$ for some positive integer k)
(iii) N_d = {n∈N/ n(x+y) = nx + ny for all x, y in N} - set of all distributive elements of N.
(iv) C(N) = {n∈N/ nx = xnfor all x in N}- centre of N.
(v) N_0 = {n∈ N/ n0 = 0} – zero -symmetric part of N.

3. PRELIMINARY RESULTS

We freely make use of the following results and designate them as R(1), R(2),...etc

R(1)N has no non – zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N. (Problem 14, p.9 of [3]).

Corresponding author: G. Sugantha[1]
[1]Department of Mathematics, Pope's college, Sawyerpuram– 628251
E-mail: sugi.trini@gmail.com

R(2) If *f* is a mate function for N, then for every x in N, *xf(x), f(x)x∈E* and *Nx =Nf(x)x, xN= xf(x)N.* (Lemma 3.2 of [6])

R(3) If L={0} and N=N_0 then (i) xy = 0 ⇒yx = 0 for all x, y in N (ii) N has Insertion of Factors Property – IFP for short (i. e) for x, y in N, xy=0 ⇒ xny=0 for all n in N. If Nsatisfies (i) and (ii) then N is said to have (∗, IFP) (Lemma 2.3 of [6])

R(4) Any weak commutative near – ring with a left identity is pseudo commutative (i.e) xyz = zyx for all x, y, z in N. (Proposition 2.8 of [7])

R(5) N has strong IFP if and only if for all ideals I of N, and for all x, y, n∈N, xy∈ I⇒ xny ∈ I (Proposition 9.2,p. 289 of [2]).

4. β_3 AND β_4 NEAR – RINGS

In this section we define β_3 and β_4 near – rings and give certain examples of these new concepts.

Definition: 4.1 Let N be a right near– ring. If for all x, y in N, xNy =yxN (xNy = Nyx) then we say N is a β_3 near – ring (β_4 near – ring).

Examples: 4.2
(a) Let (N, +) be the Klein's four group with multiplication defined as per scheme 7, p.408 of Pilz [2]

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

This near – ring N is β_3 as well as β_4. Here, the identity function serves as a mate function.

(b) Then near – ring (N, +, ·) where (N,+) is defined on Klein's four group with N={0,a,b,c} and '·' defined as per scheme 14, p.408 of Pilz [2]

.	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	0	0
c	0	a	0	c

is neither β_3 (since aNc≠ caN) nor β_4 (since aNc≠ Nca). It is worth noting that this near – ring does not admit mate functions.

(c) Let N be an arbitrary near –ring. Let I be the ideal generated by {anb – ban′ / a, b, n, n′are in N}. The factor near – ring $\overline{N}$ = N/I is a β_3 near – ring.

(d) Let N be an arbitrary near – ring. Let I be the ideal generated by {an b – n′ba/a, b, n, n′ are in N}.The factor near – ring $\overline{N}$=N/I is a β_4 near – ring

5. β_3 near – Ring

In this section we study some of the important properties of a β_3 near – rings and give a complete characterization of such near – rings.

Proposition: 5.1 If N is a $\beta_3$near – ring, then xNx = x^2 N for all x in N.

Proof: When N is a$\beta_3$near – ring, by definition, for all x, y in N, xNy = yxN (1)

The result follows by replacing y by x in (1)

Remark: 5.2 The converse of Proposition 5.1 is not true. For example, we consider the near- ring (N, +, ·) where (N, +) is the Klein's four group {0, a, b, c} and '·' is defined as per scheme 3,p.408 of Pilz [2]

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

satisfies the condition $xNx= x^2N$ for all x in N. But it is not a β_3 near – ring. [Since $aNb \neq baN$].

Proposition: 5.3 Let N be a β_3 near – ring with identity 1. Then we have the following:
(i) N is zero symmetric.
(ii) N has strong IFP.
(iii) Every N –subgroup of N is invariant.
(iv) Every left N–subgroup of N is an N - subgroup if $N = N_d$.

Proof: Let N be a $\beta_3$near – ring. Then for all x, y in N, xNy = yxN (1)

(i) Putting x=1 in (1), we get, 1Ny =y1N for all y in N. When y=0, N0 =0N= {0}. It follows that N is zero-symmetric.

(ii) Let I be an ideal of N (2)

and let xy∈ I.Now, y1x∈yNx [since 1∈N] = xyN [by (1)] ∈ IN⊆I[by (2)].

Therefore, yx∈I (3)

Now, for any n in N, we have xny∈xNy = yxN [by (1)]∈IN ⊆ I [by(2)]. From R(5), it follows that N has strong IFP.

(iii) Let S be any N-subgroup of N. Then S =$\sum_{x\in s}$ Nx (4)

Now, NxN = Nx1N = N1Nx [by (1)] = NNx⊂Nx⇒ NxN ⊂ Nx (5)

Therefore, SN = [$\sum_{x\in s}$ Nx].N [by (4)] ⊂ $\sum_{x\in s}$ NxN ⊂ $\sum_{x\in s}$ Nx [by(5)] = S[by(4)].

Consequently S is an invariant N-subgroup.

(iv) Let S be any left N-subgroup of N where N = N_d. Then S = $\sum_{x\in s}$ xN (6)

Now, NxN = 1NxN = x1NN [by (1)] = xNN⊂ xN [since SN⊂ S] ⇒ NxN ⊂ xN (7)

Hence NS= N[$\sum_{x\in s}$ xN][by(6)]⊂ $\sum_{x\in s}$ NxN ⊂ $\sum_{x\in s}$ xN[by(7)] = S[by(6)]. Consequently, every left N – subgroup is an N – subgroup.

Proposition: 5.4 Any homomorphic image of a β_3 near–ring is also a β_3 near–ring.

Proof: Straight forward.

Theorem: 5.5 Every β_3 near – ring N is isomorphic to a subdirect product of subdirectly irreducible β_3 near – rings.

Proof: By Theorem 1.62, p. 26 of Pilz [2], N is isomorphic to a subdirect product of subdirectly irreducible near–rings N_i's and each N_i is a homomorphic image of N under the projection map π_i. The rest of the proof is taken care of by Proposition 5.4.

We furnish below a necessary and sufficient condition for a β_3 near – ring to admit mate functions.

Proposition: 5.6 Let N be the β_3 near - ring. Then N admits mate functions if and only if x∈ x^2N for all x in N.

Proof: We first observe from Proposition 5.2 that, since N is β_3, xNx=x^2N for all x in N (1)

For the 'only if part', we assume that *f* is a the mate function for N. Then for all x in N, $x=xf(x)\,x\in xNx$. It follows that $x \in x^2N$.

For the 'if part 'let x∈x^2N for all x in N. Appealing to (1) we get x=xnx for some n in N. By setting n=*f*(x), we see that *f* is a mate function for N.

In the following results we assume that N has a mate function.

Theorem: 5.7 Let $N=N_d$ be a zero – symmetric β_3 near – ring with a mate function. Then we have,

(i) L={0}
(ii) N has (*, IFP)
(iii) E⊂C(N)
(iv) xN∩yN=yNxN = xyN for all x, y in N.

Proof: (i) Since *f* is a mate function for N, Proposition 5.6 demands that x∈ x^2N for all x in N. Therefore, $x=x^2n$ for some n in N. Suppose $x^2 = 0$. Clearly then x = 0. Now, R(1) guarantees that L={0}.

(ii) By (i) L= {0}. Now, R(3) guarantees that N has (∗,IFP).

(iii) Let e ∈ E. Since N is β_3, eNe = eeN = eN. Therefore, for any n in N, ene = eu and en = eve for some u, v in N.

Now, ene = (eu)e and (en)e = eve. Thus ene = en for all n in N (1)

We also have, e(ne-ene) = 0 [since $N=N_d$] ⇒ ene(ne-ene) = 0[by(ii)].And ne(ne-ene) = n.0 = 0 [Since N = N_o].

Consequently, $(ne-ene)^2 = 0$ and (i) guarantees ne-ene = 0. Therefore, ene = ne for all n in N (2)

Combining (1) and (2) we get en = ne for all n in N. Thus E⊂C(N).

(iv) First we show that for any left N-subgroups A and B of N, A∩B = BA. By Proposition 5.3 (iv), A and B are N-subgroups of N. Now, for x∈A and y∈B, yx∈BN⊂B.Therefore, BA⊂ B (3)

Also, yx∈NA⊂ A. Hence BA⊂A (4)

Combining (3) and (4), BA⊂A∩B (5)

On the other hand, if z∈ A∩B then since '*f*' is a mate function for N, z= zf(z)z∈(BN)A ⊂ BA.

Consequently, A∩B ⊂ BA (6)

Combining (5) and (6) A∩B = BA for all left N–subgroups A, B of N. We know that, xN and yN are left N- subgroups of N.

Therefore, xN∩yN=yNxN for all x, y in N. (7)

On the other hand, if y ∈ N, then, Since *f* is a mate function for N, yN = (yf(y)y)N ⊂ yNN ⇒xyN⊂xyNN = yNxN [since N is β_3].

Therefore, xyN⊂ yNxN (8)

For the reverse inclusion, yNxN = xyNN [since N is β_3]⊂ xyN. Therefore, yNxN⊂xyN (9)

From (8) and (9), yNxN = xyN (10)

for all x, y in N. Combining (7) and (10) we get xN∩yN = yNxN = xyN for all x, y in N.

We furnish below a characterization theorem for $\beta_3$near –ring.

Theorem: 5.8 Let $N=N_d$ be a zero–symmetric near-ring with a mate function *f*. Then N isβ_3 if and only if *xyN = yxN* for all x, y in N and *E⊂C(N).*

Proof: For the 'only if part', first we observe that *E⊂C(N)* (1)

[by Theorem 5.7 (iii)]. Since *f* is a mate function for N. Now, *xyN = xN∩y N* [by Theorem 5.7(iv)] = *yN∩xN = yxN* [by Theorem 5.7(iv)].

For the 'if part', first we show that' f 'is a P_3 mate function for any $x\in N$ we have $x=xf(x)x=x^2f(x)$ [Since $E\subset C(N)$]$\Rightarrow x(f(x)x - xf(x))=0 \Rightarrow xf(x)(f(x)x - xf(x))=0$ [by Theorem 5.7(ii)] and $f(x)x(f(x)x - xf(x))=f(x).0=0$ [since $N=N_0$]. Consequently, $(f(x)x - xf(x))^2 = 0$ and hence $xf(x) = f(x)x$ ……….…....(7) for all x in N [by R(1)].

Hence f is a P_3matefunction.Now, $xNy = xNf(y)y = xNyf(y)$[by (7)]$=x[yf(y)N]$[Since $E\subset C(N)$] $= xyN = yxN$ [by hypothesis].

6. β_4 near – ring

Throughout this section N denotes a β_4 near–ring. In this section we study some of the important properties.

Proposition: 6.1 If N is a β_4 near – ring, then $xNx = Nx^2$ for all x in N.

Proof: When N is a β_4 near-ring, by definition for all x, y in N, xNy = Nyx (1)

The result follows by replacing y by x in (1).

Remark: 6.2 The converse of Proposition 6.1 is not true. For example, Consider the near – ring (N, +, ·) where (N, +) is the Klein's four group {0, a, b, c} and '·' is defined as per scheme 13, p.408 of Pilz [2]

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Satisfies the condition $xNx=Nx^2$for all x in N. But it is not a β_4 near – ring [since aNb≠Nba].

Proposition: 6.3 If N is a β_4 near – ring, then NxNy = NyNx for all x, y in N.

Proof: Since N is a β_4 near – ring, we have xNy = Nyx (1)

for all x, y in N. Now, for any x, y, n in N, (nx) Ny⊂ Nx Ny ⇒ Ny(nx) ⊂ NxNy[by(1)]⇒ NyNx ⊂NxNy (2)

On the other hand, (ny) Nx⊂ NyNx⇒Nx(ny) ⊂ NyNx [by(1)] ⇒NxNy⊂ NyNx (3)

Combining (2) and (3), NxNy=NyNx for all x, y in N.

Remark: 6.4 The converse of Proposition 6.4 is not valid. For example, the near– ring (N, +, ·) where (N, +) is the Klein's four group {0, a, b, c} and '·' is defined as per scheme 20, p.408 of Pilz [2]

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

satisfies the condition NxNy = NyNx for all x, y in N. But it is not a β_4 near – ring [since 0Na≠Na0]

Theorem: 6.5 Every weak commutative near – ring with identity is β_4.

Proof: Let N be a weak commutative near – ring (1)

Let x, y∈N. For any n∈N, xny = xyn[by(1)] = nyx [byR(4)] ∈Nyx. Thus xNy⊂Nyx
(2)

On the other hand, for n_1 in N, $n_1yx = xyn_1$ [by R(4)] = xn_1y[by(1)] ∈ xNy.Consequently, Nyx⊂ xNy (3)

Combining (2) and (3) we get, N is a β_4 near – ring.

Proposition: 6.6 In a β_4 near – ring, if $N=N_d$ then xM=Mx for all N - subgroups M of N.

Proof: Let N be a β_4 near – ring. Then for any x, y in N, xNy = Nyx (1)

Let $M=\sum_{y\in M} Ny$ be an N - subgroup of N. (2)

Then $xM = x\sum_{y\in M} Ny = \sum_{y\in M} x(Ny) = \sum_{y\in M} Nyx$ [by(1)] $= [\sum_{y\in M} Ny]x = Mx$ [by(2)].

In view of Theorem 5.7 it is worth mentioning that a β_4 near – ring also possesses certain properties which are satisfied by a β_3 near – ring as in the following result.

Theorem: 6.7 Let N be a zero-symmetric β_4 near – ring with mate function. Then we have,
(i) L={0}
(ii) N has (∗, IFP)
(iii) E⊂C(N)
(iv) Nx∩Ny = NyNx= Nxy for all x, y in N.

Proof:
(i) Since f is a mate function for N, we have $x=xf(x)\ x\in xNx$. It follows that $x\in Nx^2$ [by Proposition 6.1]. Therefore, $x=nx^2$ for some n in N. Suppose $x^2 = 0$. Clearly, then $x=0$. Now, R(1) guarantees that L={0}.

(ii) By (i) L={0}. Now, R(3) guarantees that N has (∗,IFP).

(iii) Let e∈E. Since N isβ_4, eNe = Nee = Ne. Therefore for any n in N, ene = ue and ne = eve for some u, v in N.

Now, ene = eue and ene = eve. Thus ene = ne for all n in N (1)

We also have, (en – ene)e = 0⇒e(en – ene)=0 [by(ii)]⇒ene(en – ene)= 0[by(ii)].

Consequently, $(en - ene)^2 = 0$ and (i) guarantees ene=en for all n in N (2)

Combining (1) and (2) we get ne=en for all n in N. Thus E⊂C(N).

(iv) The result follows by replacing the left N–subgroups xN, yN in the proof of Theorem 5.7(iv) by the right N–subgroups Nx, Ny respectively.

We furnish below a characterization theorem for $\beta_4$near – ring.

Theorem: 6.8 Let N be a zero – symmetric near – ring with a mate function f. Then N is β_4 if and only if Nxy = Nyx for all x, y in N and E⊂C(N).

Proof: The proof is similar to that of Theorem 5.8.

REFERENCES

[1] J. R. Clay, The near – rings on groups of low order, Math.Z.104 (1968), 364 – 371

[2] Gunter Pilz, near – rings, North Holland, Amsterdam, 1983

[3] N. H. MC. Coy, The Theory of Rings, Macmillan & Co, 1970

[4] G. Sugantha and R. Balakrishnan, β_1 Near-ring, to appear in International Journal of Algebra, Vol. 8, 2014. No.1, 1-7

[5] G. Sugantha and R. Balakrishnan, β_2 Near-rings, to appear in International Journal of Ultra Scientist of Physical Sciences, Vol. 26, No (1), 2014 April.

[6] S. Suryanarayanan and N. Ganesan, Stable and Pseudo Stable Near – Rings, Indian J. Pure and Appl.Math19 (12) December, 1988, 1206 – 1216.

[7] S. Uma, Pseudo Commutative Near – Rings, Scientia Magna, An international Journal, Vol.6 (2010) No.2,75 – 85.

Source of Support: Nil, Conflict of interest: None Declared

