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THE ELEMENT IDEAL GRAPHS OF A RING OF INTEGERS

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ABSTRACT

Let R be a commutative ring with identity and let x be an element of R. The Element Ideal Graph $\Gamma_x(R)$ is a graph whose vertex set is the set of nontrivial ideals of R and two vertices I and J are adjacent if and only if $x \in I\,J$. In this paper we consider the element ideal graph of the ring of integers.

Keywords: *Zero divisor graph, annihilating ideal graph and element ideal graph.*

2010 Mathematics subject classification: *05C25.*

INTRODUCTION

Let R be a commutative ring with identity, and let Z(R) be its set of zero divisors. We associate a simple graph Γ(R) to R with vertices Z*(R) = Z(R)\ {(0)}, the set of all non-zero zero divisors of R, and for distinct x, y ∈Z*(R), the vertices x and y are adjacent if and only if xy=0. Obviously Γ(R) is empty if R is an integral domain.

The zero divisor graph of a commutative ring was introduced by Beck in [4], and further studied in [1, 2, 3, 9, 10]. The annihilating ideal graph AG(R) is a graph with vertex set AG*(R) =AG(R)\ {(0)} such that there is an edge between vertices I and J if and only if $I\,J = (0)$. The idea of annihilating ideal graph was introduced by Behboodi and Rakeei in [5, 6].

In [11], we introduced the notion of the element ideal graph of a commutative ring. In the present paper we consider the element ideal graph of the ring of integers.

From now on we shall use the symbolI— Jto denote for two adjacent ideal vertices I and J, and we use Z to denote the set of integer numbers.

1. BACKGROUND

In this section we state some definitions and theorems that we need in our work.

Definition: 1.1[11, P.404] Let R be a commutative ring with identity and let x∈R. The element ideal graph is a graph whose vertex set is nontrivial ideals of R, and two of its vertices I and J are adjacent if and only if x ∈ I J. We denote the element ideal graph by $\Gamma_x(R)$.

Definition: 1.2[8]

1. The distance d(u, v) between a pair of vertices u and v of the graph Γ is the minimum of the lengths of the u—v paths ofΓ.
2. The degree of the vertex *a* in the graph Γ is the number of edges incident to *a*.
3. The diameter of the graph Γ is the maximum distance between any two distinct vertices.
4. The girth of the graph Γ is the length of the shortest cycle in Γ.
5. A bipartite graph is one whose vertex set is partitioned into two disjoint subsets in such a way that the two end vertices for each edge lie in distinct partition. The complete bipartite graph with exactly two partitions of order m and 1 is called star.

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6. A complete subgraph K_n of a graph Γ is called a clique, and cl(Γ) is the clique number of Γ, which is the greatest integer r≥ 1 such that $K_r \subseteq \Gamma$.
7. The graph Γ is called a plane graph if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all. A graph which is isomorphic to a plane graph is called a planar graph.

Theorem: 1.3[8, P.96] (Kuratowsky Theorem) A graph Γ is planar if and only if it does not contains a graph homomorphic with K_5 or K (3, 3).

Theorem: 1.4[5, P.8] For every ring R, the annihilating-ideal graph AG(R) is connected and diam(AG(R))≤3. Moreover, if AG(R) contains a cycle, then gr (AG(R)) ≤ 4.

Theorem: 1.5[12] Let n>1 be a non-prime integer. Then, $\Gamma(Z_n)$ contains a subgraph which isomorphic with $AG(Z_n)$.

2. THE ELEMENT IDEAL GRAPH OF A RING OF INTEGERS

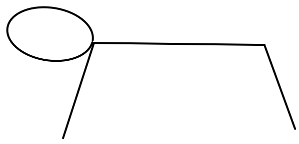
The main purpose of this section is to investigate the element ideal graph of the ring of integers.

We begin this section by the following example.

Example: 1 The graph $\Gamma_{12}(Z)$ is:

(2) (3)

(6) (4)



Lemma: 2.1 A nontrivial ideal I of Z is an ideal vertex of $\Gamma_x(Z)$ if and only if *x* divisible by the generator of I.

Proof: Let I= (*a*) be an ideal vertex of $\Gamma_x(Z)$. Then there exists an ideal vertex (b) of $\Gamma_x(Z)$ such that x∈(*a*)(b). This implies that there exists an integer r such that x=r*a*b. Hence *a*|x.

Conversely, if x is divisible by the generator of I= (*a*), then there exists an integer r such that x=r*a*. Obviously x∈(*a*)(r). Hence (*a*) is an ideal vertex of $\Gamma_x(Z)$.

Example: 2 obviously (2) is an ideal vertex of $\Gamma_{12}(Z)$ and 2 divides 12.

Proposition: 2.2 Let x∈Z^+-{1}.Then x is a prime number if and only if $\Gamma_x(Z)=\emptyset$.

Proof: Let $\Gamma_x(Z)=\emptyset$. Then by Lemma2.1, the only divisors of x are ∓1 and ∓x. This means that x is a prime number.

Conversely, ifx is a prime number, then x has no divisor except ∓1 and ∓x. Then by Lemma2.1, $\Gamma_x(Z)=\emptyset$.

Example: 3 The graph $\Gamma_2(Z)$ is an empty graph.

The next result illustrates the adjacency of two ideals in the element ideal graph of the ring of integers.

Theorem: 2.3 Let x, *a*, *b*∈Z-{0,∓1}. If (*a*) and (b) are ideal vertices of $\Gamma_x(Z)$ such that *a* and b are relatively prime integers, then (*a*) and (b) are adjacent ideal vertices in $\Gamma_x(Z)$.

Proof: Since (*a*) and (b) are ideal vertices of $\Gamma_x(Z)$, then by Lemma 2.1, both *a* and b divide x. Since the common divisor of *a*and b is equal to 1, then *a*b divides x. This means that x∈(*a*)(b). Thus (*a*) and (b) are adjacent in$\Gamma_x(Z)$.

Example: 4 The ideals (3) and (4) of Z are adjacent ideal vertices in $\Gamma_{12}(Z)$, since 3 and 4 are relatively prime.

The converse of Theorem2.3 may not be true in general, as the following example shows.

Example: 5 The ideals (6) and (4) of Z are adjacent ideal vertices in $\Gamma_{24}(Z)$, while 6 and 4 are not relatively prime.

The next result illustrates that $\Gamma_x(Z)$ is infinite if and only if x=0.

Proposition: 2.4 Every nontrivial ideal of Z is a vertex of $\Gamma_x(Z)$ if and only if x=0.

Proof: Suppose that every ideal (*a*) of Z is an ideal vertex of $\Gamma_x(Z)$.Then by Lemma2.1, every nonzero integer divides x. This statement is true for the only when x=0.

The next result demonstrates that the divisibility leads to more adjacency in the element ideal graph.

Theorem: 2.5 Let x, a, $b\in$Z-$\{0,\mp1\}$with b is a non-prime integer. Then (a) is adjacent to (b) in $\Gamma_x(\mathrm{Z})$ if and only if (a) is adjacent to all non-trivial ideals which are generated by divisors of b.

Proof: Let (a) and (b) be two adjacent ideal vertices of $\Gamma_x(\mathrm{Z})$ and let c$\neq \mp1$ be the divisor of b. This means that x∈(a)(b) and c|b. Then there exists r,s∈Z such that x=rab and b=sc. This implies that x=rs$a$c. Thus x∈(a)(c). This means that (a) and (c) are adjacent ideal vertices in $\Gamma_x(\mathrm{Z})$.

Conversely, if (a) is adjacent to all nontrivial ideals which are generated by divisors of b, then (a) is adjacent to (b), since b is a divisor of itself.

Example: 6 The ideals (2) and (6) of Z are adjacent in $\Gamma_{12}(\mathrm{Z})$ and 3 is a divisor of 6. So (2) is also adjacent to (3) in $\Gamma_{12}(Z)$.

From Theorem2.5 the following corollary is immediate.

Corollary: 2.6 Let x, a, $b\in$Z-$\{0,\mp1\}$. If $\Gamma_x(\mathrm{Z})$ consists of only one edge (a) — (b) of distinct terminals, then b is either a prime number or divisible by a.

Proof: Let b is a non-prime number. Then there exists, $c\in$Z-$\{0,\mp1, \mp\mathrm{b}\}$such that c|b. By Theorem2.5, (a) and (c) are adjacent $\Gamma_x(Z)$. Since $c\neq\mp$b and $\Gamma_x(\mathrm{Z})$ consists of the only one edge (a) — (b), then (c) = (a). This implies that a=∓c. Therefore a|b.

Example: 7 The graph $\Gamma_{27}(\mathrm{Z})$ consists of the edge (3) — (9) and the loop (3) — (3), and 9 is divisible by 3.

In the next result, we put a certain condition for the element ideal graph to be a star graph.

Theorem: 2.7 Let x be a nonzero integer such that $\Gamma_x(\mathrm{Z}) \neq \emptyset$. Then $\Gamma_x(\mathrm{Z})$ is a star graph if and only if $\Gamma_x(\mathrm{Z})$ consists of only one edge of distinct terminals.

Proof: Let $\Gamma_x(\mathrm{Z})$ be a star graph with center (a) , and let (b) and (c) be two ideal vertices incident to (a) in $\Gamma_x(\mathrm{Z})$. This means that x∈(a)(b)∩(a)(c). Then there exist s, r∈ x∈Z such that x=rab=sac. This implies that x∈(ra)(b)∩(sa)(c). Thus (b) and (c) are adjacent to (ra) and (sa) respectively. But (b) and (c) are the end vertices of $\Gamma_x(\mathrm{Z})$, so (a)=(ra)=(sa).This gives that r=∓1 and s=∓1. Since rab=sac, then ∓ab=∓ac. The cancellation law gives∓b=∓c. Thus (b)=(c) . Hence $\Gamma_x(\mathrm{Z})$ consists of the only one edge of distinct terminals (a) and (b).

The converse is clear, since every graph consisting of one edge is a star graph.

Example: 8 The graph $\Gamma_{15}(\mathrm{Z})$ is a star graph with the only edge (3)—(5).

The following result shows that the graph $\Gamma_x(\mathrm{Z})$may be a star graph of looped center.

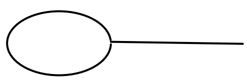
Theorem: 2.8 Let n∈Z^+-$\{1\}$ and let p be a prime number. Then the graph$\Gamma_{p^n}(\mathrm{Z})$ is star graph of looped center (p) if and only if either n=2 or n=3.

Proof: If n=2, then $\Gamma_{p^n}(\mathrm{Z})$consists of the loop (p)— (p). If n=3, then $\Gamma_{p^n}(\mathrm{Z})$consists of the edge (p)— (p^2) and the loop (p)— (p). From both cases we see that the graph $\Gamma_{p^n}(\mathrm{Z})$ is a star graph of looped center (p).

Conversely, suppose that $\Gamma_{p^n}(\mathrm{Z})$ is a star graph of looped center (p). Since $\Gamma_{p^n}(\mathrm{Z}) \neq \emptyset$, then by Proposition2.2, n$\neq$ 1. Now we determine those integers at which $\Gamma_{p^n}(\mathrm{Z})$ is a star graph of looped center (p). If n=4, then $\Gamma_{p^n}(\mathrm{Z})$ consists of the edges (p)— (p^2) and (p)— (p^3) with the loops (p)— (p) and (p^2)— (p^2).In this case $\Gamma_{p^n}(\mathrm{Z})$ is not a star graph, because it has a loop at the vertex (p^2). If n>4, then $p^n\in(p)(p^2)\cap(p^2)(p^3)\cap(p)(p^3)$. This means that (p)—($p^2$)—($p^3$)—(p) is a cycle in $\Gamma_{p^n}(\mathrm{Z})$. In this case $\Gamma_{p^n}(\mathrm{Z})$ is not a star graph for every n>4. Hence the only cases for $\Gamma_{p^n}(\mathrm{Z})$ to be a star graph of looped center (p) are n=2 and n=3.

Example: 9 The graph $\Gamma_8(\mathrm{Z})$ is a star graph of looped center (2).

(2) (4)



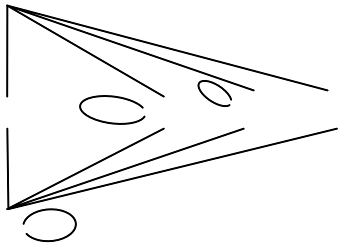
The next result illustrates the planarity of the graph $\Gamma_{p^n}(Z)$.

Theorem: 2.9 Let $n\in Z^{+}$-{1} and let p be a prime number. Then the graph$\Gamma_{p^n}(Z)$ is planar graph if and only if n<9.

Proof: Suppose that n<9. If n=8, then the graph$\Gamma_{p^n}(Z)$ can be constructed as follows:

(p^2)

$(p^6)(p^4)$— (p^3)—(p^5)



(p^7)— (p)

Clearly the graph $\Gamma_{p^8}(Z)$ is a planar graph. Since$\Gamma_{p^2}(Z)$, $\Gamma_{p^3}(Z)$,…, $\Gamma_{p^7}(Z)$ are subgraphs of $\Gamma_{p^8}(Z)$, then they are also planar graphs.

Conversely, suppose that $\Gamma_{p^n}(Z)$ is a planar graph. We have to show that n<9. If n≥9, then the graph $\Gamma_{p^n}(Z)$contains a complete subgraph K_5 whose vertices are (p) ,(p^2),(p^3),(p^4) and (p^5). Then by Kuratowsky Theorem in [7], $\Gamma_{p^n}(Z)$ is not planar graph. This contradicts the fact that $\Gamma_{p^n}(Z)$ is a planar graph. Therefore n must be less than 9.

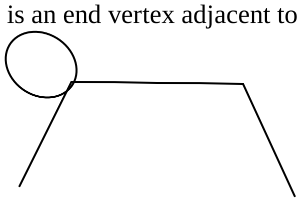
Proposition: 2.10 Let x, a, $b\in Z$-{0,∓1} such that (a) is an end vertex of $\Gamma_x(Z)$. Then (a)—(b) is an edge of $\Gamma_x(Z)$ iff x=∓ a b.

Proof: Let (a)— (b) be an edge of $\Gamma_x(Z)$. Then x∈(a)(b) . This implies that x=rab for some r∈Z, then x∈(a)(rb).This means that (a)—(rb) is an edge of $\Gamma_x(Z)$. Since (a) is an end vertex of$\Gamma_x(Z)$, then (b) = (rb). This gives that r=∓1, and hence x=∓ab.

The converse is clear, since x=∓a b implies that x∈(a)(b).This means that (a)—(b) is an edge in$\Gamma_x(Z)$.

Example: 10 In the graph $\Gamma_{12}(Z)$, the ideal vertex (6) is an end vertex adjacent to the ideal vertex (2).

(2) (3)



(6) (4)

$\Gamma_{12}(Z)$

In the next result we find the clique number of $\Gamma_{p_1p_2\cdots p_n}(Z)$.

Theorem: 2.11 If $n\in Z^{+}$-{1} and p_1, p_2 ,… ,p_n are distinct prime numbers, then the graph $\Gamma_{p_1p_2\cdots p_n}(Z)$ contains a maximal complete subgraph of order n, moreover cl($\Gamma_{p_1p_2\cdots p_n}(Z)$)=n.

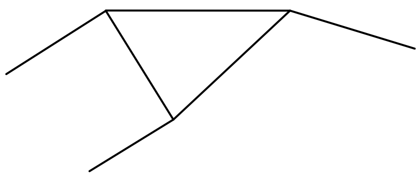
Proof: Define the graph G by G={(p_i)—(p_j):i,j=1,2,…,n}. Since $p_1p_2\dots p_n\in(p_i)(p_k)$for every i, k=1, 2,…,n. Obviously, G is a complete subgraph of $\Gamma_{p_1p_2\cdots p_n}(Z)$ of order n. To show that the graph G is a maximal complete subgraph of$\Gamma_{p_1p_2\cdots p_n}(Z)$, let($a$) be any ideal vertex of $\Gamma_{p_1p_2\cdots p_n}(Z)$ different from$(p_1),(p_2),\dots,(p_n)$ and adjacent to all of them. Then at least one of $p_1,p_2,\dots,p_n$, say p_1 is different from $a$and divides it. Since (a) is adjacent to (p_1), then $p_1p_2\dots p_n\in(p_1)(a)$. This implies that there exists an integer r such that $p_1p_2\dots p_n=rp_1a$. Since $p_1|a$, then $p_1^2|p_1a$. This implies that $p_1^2|p_1p_2\dots p_n$. This contradicts the fact that p_1 , p_2 ,… ,p_n are distinct prime numbers. Therefore G is a maximal complete subgraph of $\Gamma_{p_1p_2\cdots p_n}(Z)$. Hence cl($\Gamma_{p_1p_2\cdots p_n}(Z)$)=n.

Example: 11 Consider the graph$\Gamma_{30}(Z)$.

(2) (3)

(10)
(15)

(5)
(6)



$\Gamma_{30}(Z)$

Clearly cl(Γ_{30}(Z) is equal to the number of primes which are divide 30, that is cl(Γ_{30}(Z)=3.

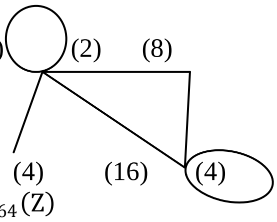
In the next result we find the clique number of Γ_{p^n}(Z).

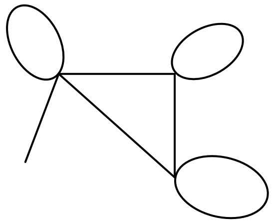
Theorem: 2.12 Let p be a prime number. Then $\Gamma_{p^{2n+1}}$(Z))=$\Gamma_{p^{2n+2}}$(Z))=n+1 for every n∈Z^+.

Proof: Clearly, the ideal vertices of $\Gamma_{p^{2n+1}}$(Z) and $\Gamma_{p^{2n+2}}$(Z) are (p), (p^2),…,(p^{2n}) and (p), (p^2), … , (p^{2n+1}) respectively. Since p^{2n}∈(p^i)(p^j) , for all i,j=1,2,…,n+1, then the graph G={(p^i)—(p^j): i, j=1,2,…,n+1} is a complete subgraph of $\Gamma_{p^{2n+1}}$(Z) and $\Gamma_{p^{2n+2}}$(Z).To show that G is a maximal complete subgraph of $\Gamma_{p^{2n+1}}$(Z), let (p^m) be any ideal vertex of $\Gamma_{p^{2n+1}}$(Z) such that m>n+1, then p^{2n+1}∉(p^m)(p^{n+1}), which means that (p^m) is not adjacent to (p^{n+1}). Thus G is a maximal complete subgraph of $\Gamma_{p^{2n+1}}$(Z) of order k+1, and hence cl($\Gamma_{p^{2n+1}}$(Z))=n+1 for every n∈Z^+. It is remain to show that G is also a maximal complete subgraph of $\Gamma_{p^{2n+2}}$(Z). If (p^m) is an ideal vertex of $\Gamma_{p^{2n+2}}$(Z) such that m>n+1, then p^{2n+2}∉(p^m) (p^{n+1}), that means (p^m) is not adjacent to (p^{n+1}). Thus G is a maximal complete subgraph of $\Gamma_{p^{2n+2}}$(Z) of order k+1. Hence cl($\Gamma_{p^{2n+2}}$(Z))=n+1for every n∈Z^+.

Example: 12 Consider the graphs Γ_{32}(Z) and Γ_{64}(Z).

(2) (8) (2) (8)





(16) (4) (16) (4)
Γ_{32}(Z) Γ_{64}(Z)

Clearly cl(Γ_{32}(Z))=cl(Γ_{64}(Z))=3.

3. THE RELATIONSHIPΓ_n(Z) AND AG (Z_n)

In this section, the relationship between two element ideal graphs of the ring of integers will be explored. Moreover the relationship between the element ideal graph and the graph of annihilating ideals will be illustrated.

We start this section with the following result.

Proposition: 3.1 Let xy∈Z^+\{1} such that x is a product of two relatively prime integers. Then x|y if and only if $\Gamma_x(Z) \subseteq \Gamma_y(Z)$.

Proof: Let x|y. Then x is a factor of y. So by Proposition2.14 in [10], $\Gamma_x(Z) \subseteq \Gamma_y(Z)$.

Conversely, let$\Gamma_x(Z) \subseteq \Gamma_y(Z)$, and let x=*a*b for some relatively prime integers *a*b∈Z^+\{1}. Then x∈(*a*)(b). This means that (*a*)— (b) is an edge in Γ_x(Z). Since $\Gamma_x(Z) \subseteq \Gamma_y(Z)$, then (*a*)— (b) is also an edge in Γ_y(Z). By Lemma2.1, both *a* and b divide y. Since *a* and b are relatively prime integers, then x=ab|y.

Example: 13 The graph Γ_6(Z) is a subgraph of Γ_{12}(Z) , since6 divides 12**.**

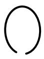
The next result gives a sufficient condition for two element ideal graphs to be disjoint.

Proposition: 3.2 If x and y are relatively prime integers, then $\Gamma_x(Z)$ and Γ_y(Z) are disjoint.

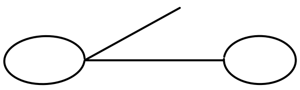
Proof: Let $\Gamma_x(Z) \cap \Gamma_y(Z) \neq \emptyset$. Then Γ_x(Z) and Γ_y(Z) contain an edge say (*a*)—(b) .From Lemma2.1, both *a* and b divide x and y. Since gcd(x, y) =1, then 1=*a*b∈(*a*). This contradicts the fact that (*a*) is a nontrivial ideal. Therefore Γ_x(Z) and Γ_y(Z) are disjoint.

Example: 14 The graphs Γ_9(Z) and Γ_{16}(Z) are disjoint, since 9 and 16 are relatively prime integers.

(8)



(3) (2) (4)



Γ_9(Z)Γ_{16}(Z)

Clearly $\Gamma_9(Z) \cap \Gamma_{16}(Z) = \emptyset$.

The next example shows that the converse of Proposition3.2 may not be true in general.

Example: 15 The graphs $\Gamma_4(Z)$ and $\Gamma_6(Z)$ are disjoint, while 4 and 6 are not relatively prime integers.

The next result gives a sufficient condition for two element ideal graphs to be identical.

Theorem: 3.3 Let x, y∈Z-{0,$\pm$1} be non-primes. Then $\Gamma_x(Z)=\Gamma_y(Z)$ if and only if x=y.

Proof: Let $\Gamma_x(Z)=\Gamma_y(Z)$. Then x, y∈(a)(b) for every edge (a)—(b) of $\Gamma_x(Z)$ and $\Gamma_y(Z)$. Then there exist s, r∈Z such that x=rab and y=sab. This implies that x∈(ra)(b)and$y$ ∈(sa)(b), hence(ra)—(b) and (sa)—(b) are edges in$\Gamma_x(Z)=\Gamma_y(Z)$. This yields that y∈(ra)(b) and x ∈(sa)(b), it follows that there exist integers t and w such that y=trab=tx and x=wsab=wy . This implies that x|y and y|x. Thus x=y.

Example: 16 The graphs $\Gamma_4(Z)$ and $\Gamma_8(Z)$are not identical, while 4$\neq$ 8.

It is natural to ask whether $\Gamma_x(Z)$ and $\Gamma_y(Z)$ are isomorphic for every x, y∈Z-{0,$\mp$1}, the answer is negative, as the following example shows.

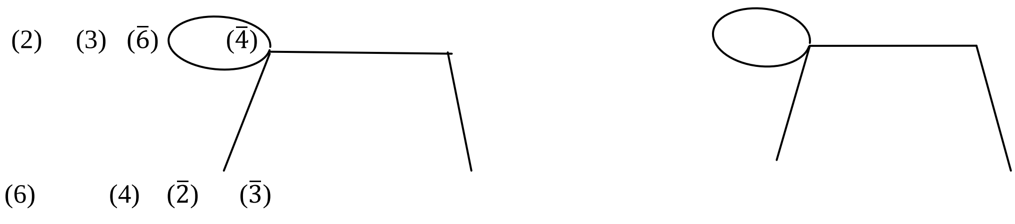
Example: 17 Clearly$\Gamma_6(Z)$and$\Gamma_{12}(Z)$are not isomorphic, since. The number of vertices of $\Gamma_6(Z)$ is equal to 2, while the number of vertices of $\Gamma_{12}(Z)$ is equal to 4.

The next result gives a condition which ensure the isomorphism between the element ideal graph of the ring of integers and the annihilating ideal graph of the ring of integers modulon.

Theorem: 4.2.4 If n∈Z+-{1} is not prime number, then $\Gamma_n(Z) \cong AG(Zn)$.

Proof: Since n>1 is not prime, then n has some divisors. We denote all positive divisors of n by α_1, α_2,...,$\alpha_m$with $\alpha_1 < \alpha_2 <...<\alpha_m$ for some m∈Z+. Since $\alpha_1, \alpha_2,...,\alpha_m$ are divisors of n , then there exist $\beta_1, \beta_2,\ldots,\beta_m$ such that n=$\alpha_k\beta_k$, for every k=1,2,…,m. By Lemma4.1.1, the only ideal vertices of $\Gamma_n(Z)$ are (α_1), (α_2),..., (α_m). It is clear that the only ideal vertices of $AG(Z_n)$ are ideals $(\overline{\alpha_1})$, $(\overline{\alpha_2})$,..., $(\overline{\alpha_m})$ of Zn. Define the mapping f:$\Gamma_n(Z)\rightarrow AG(Z_n)$ by f((α_k))= $(\overline{\beta_k})$. Obviously, f is onto. Now we give two ideal vertices (α_l) and (α_k) of $\Gamma_n(Z)$ such that f((α_l))=f((α_k)). This implies that $(\overline{\beta_l}) = (\overline{\beta_k})$. Since $\beta_l \in (\overline{\beta_l}) = (\overline{\beta_k})$, then $\beta_l - \beta_k = sn$ for some integer s. This means that $n|(\beta_l - \beta_k)$. And this statement is true only when $\alpha_l = \alpha_k$. This implies that (α_1)=(α_2). Thus f is one to one. Suppose that (α_l) and (α_k) are adjacent ideal vertices in $\Gamma_n(Z)$. This means that n∈ $(\alpha_l)(\alpha_k)$.Then there exists an integer r such that n=r$\alpha_l\alpha_k$.Since n=$\alpha_l\beta_l = \alpha_k\beta_k$, then $(\alpha_l\beta_l)(\alpha_k\beta_k)$ =$(r\alpha_l\alpha_k)(r\alpha_l\alpha_k)$.Then the cancellation gives$\beta_l\beta_k = r^2\alpha_k\alpha_l$=rn. Thus $\overline{\beta_l}\overline{\beta_k}=\overline{0}$. This implies that f($(\alpha_l)$)f($(\alpha_k)$)=$(\overline{\beta_l})(\overline{\beta_k})$=$(\overline{0})$. This means that f($(\alpha_1)$) and f($(\alpha_2)$) are adjacent ideal vertices in $AG(Z_n)$. Thus f preserves the adjacency property. Hence $\Gamma_n(Z) \cong AG(Z_n)$.

Example: 18 The graphs $\Gamma_{12}(Z)$ and $AG(Z_{12})$ can be drown as follows.



$\Gamma_{12}(Z)$ $AG(Z_{12})$

Clearly $\Gamma_{12}(Z) \cong AG(Z_{12})$**.**

The following corollaries follow from Theorem3.4.

Corollary: 3.5 If n>1 is a nonprime integer, then $\Gamma_n(Z)$ is a finite connected graph with diameter less than or equal to 3, and the girth less than or equal to 4.

Proof: From Theorem1.4, the graph $AG(Z_n)$ is a finite connected graph with diameter less than or equal to 3, and the girth less than or equal to 4. Then by Theorem3.4, the graph $\Gamma_n(Z)$ is also connected with diameter less than or equal to 3, and the girth less than or equal to 4.

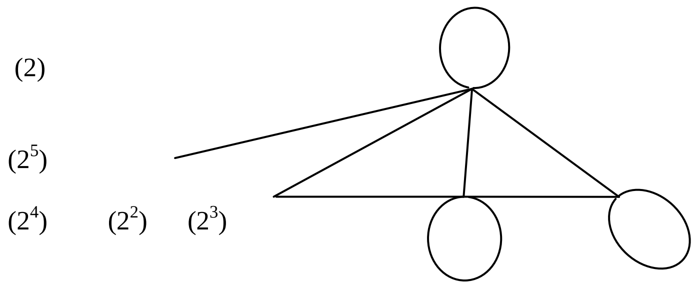
Corollary: 3.6 If $n \in Z^+ - \{1\}$ is not prime number, then $\Gamma_n(Z)$ contains a subgraph which isomorphic to $\Gamma_n(Z)$.

Proof: From Theorem1.5, there exists a subgraph G of $\Gamma_n(Z)$ such that $G \cong AG(Z_n)$. Then by Theorem 3.4, $G \cong \Gamma_n(Z)$.

Corollary: 3.7 If p is a prime number, then $\Gamma_{p^n}(Z)$ has diameter 2 for every $n \in Z^+ - \{1, 2, 3\}$.

Proof: First we prove that diam($AG(Z_{p^n})$)=2. Obviously the ideal vertices of $AG(Z_{p^n})$ are $(\overline{p^1})$, $(\overline{p^2})$,…, $(\overline{p^{n-1}})$. Let $(\overline{p^m})$ and $(\overline{p^k})$ be any two distinct ideal vertices of $AG(Z_{p^n})$. Since $(\overline{p^m})(\overline{p^{n-1}}) = (\overline{p^k})(\overline{p^{n-1}}) = (\overline{0})$, then $(\overline{p^m})$ and $(\overline{p^k})$ are adjacent to $(\overline{p^{n-1}})$ in AG(R).This means that the distance between any two distinct ideal vertices of $AG(Z_{p^n})$ is less than or equal to 2. Since the diameter is the maximum distance between any two distinct vertices, then the diameter of $AG(Z_{p^n})$ is equal to 2 for every n>3.ByTheorem3.4, diam($\Gamma_{p^n}(Z)$=2 for every $n \in Z^+ - \{1,2,3\}$.

Example: 19 Consider the element ideal graph $\Gamma_{64}(Z)$.



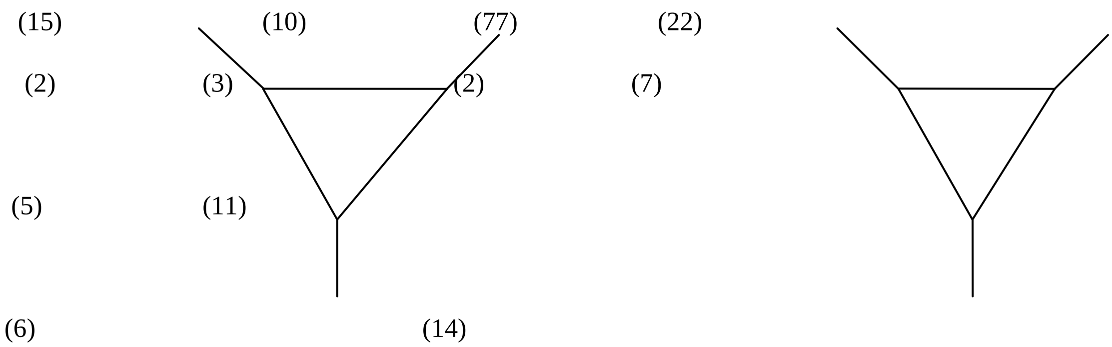
Clearly the diameter of $\Gamma_{64}(Z)$is equal to 2.

Before we close this section, we give the following result.

Theorem: 3.8 Let $n \in Z^+$ and α_1, α_2,...,α_n be non-negative fixed integers. Then all graphs of the form $\Gamma_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}(Z)$ are isomorphic, for all choices of distinct prime numbers $p_1, p_2, \dots, p_n$.

Proof: Let $p_1, p_2, \dots, p_n$, $q_1, q_2, \dots, q_n$ be prime numbers with $p_i \neq p_j$ and $q_i \neq q_j$ for all i≠j.Give two graph $\Gamma_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}(Z)$ and $\Gamma_{q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}}(Z)$. Define a mapping $f: \Gamma_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}(Z) \to \Gamma_{q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}}(Z)$ by $f((p_1^{s_1} p_2^{s_2} \dots p_n^{s_n}))=(q_1^{s_1} q_2^{s_2} \dots q_n^{s_n})$, where s_1 , s_2 ,…, s_n are non-negative integers less than or equal to α_1, α_2,...,α_n respectively. Clearly , f is onto. Let $I=(p_1^{s_1} p_2^{s_2} \dots p_n^{s_n})$ and $J=(p_1^{r_1} p_2^{r_2} \dots p_n^{r_n})$ be two ideal vertices of $\Gamma_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}(Z)$ such that f(I)=f(J). This means that $(q^{s_1} q_2^{s_2} \dots q^{s_n})=(q_1^{r_1} q_2^{r_2} \dots q_n^{r_n})$. This implies that$q^{s_1} q_2^{s_2} \dots q^{s_n} = q_1^{r_1} q_2^{r_2} \dots q_n^{r_n}$, since every two ideals of Z of distinct positive generators are distinct. It follows that $s_i = r_i$, for all i=1, 2,…,n. Thus $I=(p_1^{s_1} p_2^{s_2} \dots p_n^{s_n})=(p_1^{r_1} p_2^{r_2} \dots p_n^{r_n})=J$. Hence f is one to one. Let$I=(p_1^{s_1} p_2^{s_2} \dots p_n^{s_n})$ and $J=(p_1^{r_1} p_2^{r_2} \dots p_n^{r_n})$ be adjacent ideal vertices in$\Gamma_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}(Z)$. Then $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \in IJ$. From Lemma2.1,$p_1^{s_1} p_2^{s_2} \dots p_n^{s_n}$ and $p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$ divide $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$. Thus $s_i = r_i \leq \alpha_i$ for all i=1, 2,…,n. This implies that $q_1^{s_1} q_2^{s_2} \dots q_n^{s_n}$ and $q_1^{r_1} q_2^{r_2} \dots q_n^{r_n}$ divide $q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}$.FromLemma2.1, $q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} \in (q_1^{s_1} q_2^{s_2} \dots q_n^{s_n})(q_1^{r_1} q_2^{r_2} \dots q_n^{r_n})$=f(I)f(J). This means that f(I) and f(J) are adjacent ideal vertices in $\Gamma_{q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}}(Z)$. So f preserves the adjacency property. Hence$\Gamma_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}(Z)$ and $\Gamma_{q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}}(Z)$ are isomorphic.

Example: 20 Consider the graphs $\Gamma_{30}(Z)$ and $\Gamma_{154}(Z)$.



$\Gamma_{30}(Z) \Gamma_{154}(Z)$

Clearly, $\Gamma_{30}(Z) \cong \Gamma_{154}(Z)$.

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