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FIXED POINT AND COMMON FIXED POINT THEOREM IN BANACH SPACE TAKING RATIONAL EXPRESSION FOR 1, 2, 3 MAPPING

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ABSTRACT

In the present paper we will establish some fixed point and common fixed point theorem in Banach space taking rational expression for 1,2,3 mappings. Our result is extended form of many known results taking particular inequality.

Keywords: *Fixed point, Common fixed point Banach space.*

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1. INTRODUCTION:

In recent years, nonlinear analysis have attracted much attention .The study of non contraction mapping concerning the existence of fixed points draw attention of various authors in non linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally nonlinear, therefore fixed point methods especially Banach contraction principle provide powerful tool for obtaining the solution of these equations which are very difficult to solve by other method. Recently Verma [24] described about the application of Banach contraction principle [2].

Browder [4] was the first mathematician to study non expansive mappings. Mean while Browder [4] and Ghode [6] have independently proved a fixed point theorem for non expansive mapping.

Many other Mathematicians have done the generalization of non-expansive mappings as well as noncontract ion mappings Kirk [15, 16 & 17] gives the comprehensive survey concerning fixed point theorems for non expansive mappings.

Recently Rajesh Shrivastava, sabha Kant Dwivedi and S.S. Rajput [29] generalizes non contraction mappings.

Before start the main result we write some definitions.

2. PRELIMINARIES:

Definition 2.1: Let L be a linear space and $\|,\|$ is nonnegative, real valued function define on L such that for all $x, y \in L$ and $\alpha \in R$ or C

(i) $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|x + y\| \le \|x\| + \|y\|$

(iii) $\|\alpha x\| = |\alpha| \|x\|$

Then $\|,\|$ is called norm and ($L, \|,\|$) is called nor med linear space.

Definition 2.2: A sequence $\{x_n\}$ in a normed linear space L is called Cauchy sequence if $\lim_{x\to\infty} \|x_n - x\| = 0$

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Definition 2.3: A sequence $\{x_n\}$ in a normed linear space L, is called sequence if $\lim_{x\to\infty}\|x_n - x_m\| = 0$

Definition 2.4: A normed linear space in which every Cauchy sequence is convergent is called Banach space.

3. MAIN RESULT:

Theorem 3.1:-Let F be a mapping of a Banach space X into it self. If F satisfies the following conditions;

$F^2 = I$, where I is identity mapping. (3.1.1)

$$\|F(X)-F(Y)\| \tag{3.1.2}$$

$$\leq \alpha\left[\frac{\|X-F(X)\|\|X-Y\|+\|Y-F(Y)\|\|Y-F(X)\|+\|X-Y\|^2}{\|X-F(X)\|+\|X-Y\|}\right]$$

$$+\beta\left[\frac{\|Y-F(Y)\|\|X-Y\|+\|X-F(X)\|\|X-F(Y)\|+\|X-Y\|^2}{\|Y-F(Y)\|+\|X-Y\|}\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\left[\|X-F(Y)\|+\|Y-F(X)\|\right]+\eta\|X-Y\|$$

For every $x,y\in X$, where $\alpha, \beta, \gamma, \delta, \eta > 0$ and $5\alpha + 5\beta + 4\gamma + 2\delta + \eta < 2$, then F has a fixed point.

If $\alpha + \beta + 2\delta + \eta < 1$. Then F has a unique fixed point.

Proof: Suppose X is a point in the Banach space X,

Taking $Y = \frac{1}{2}(F+I)(X)$, $Z = F(Y)$ and $u = 2Y - Z$ we have

$$\|Z-X\| = \|F(Y)-F^2(X)\| = \|F(Y)-F(F(X))\|$$

$$\leq \alpha\left[\frac{\|Y-F(Y)\|\|Y-F(X)\|+\|F(X)-F^2(X)\|\|F(X)-F(Y)\|+\|Y-F(X)\|^2}{\|Y-F(Y)\|+\|Y-F(X)\|}\right]$$

$$+\beta\left[\frac{\|F(X)-F^2(X)\|\|Y-F(X)\|+\|Y-F(Y)\|\|Y-F^2(X)\|+\|Y-F(X)\|^2}{\|F(X)-F^2(X)\|+\|Y-F(X)\|}\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-F^2(X)\|\right]+\delta\left[\|Y-F^2(X)\|+\|F(X)-F(Y)\|\right]+\eta\left[\|Y-F(X)\|\right]$$

$$=\alpha\left[\frac{\|Y-F(Y)\|\|Y-F(X)\|+\|F(X)-X\|\|F(X)-F(Y)\|+\|Y-F(X)\|^2}{\|Y-F(Y)\|+\|Y-F(X)\|}\right]$$

$$+\beta\left[\frac{\|F(X)-X\|\|Y-F(X)\|+\|Y-F(Y)\|\|Y-X\|+\|Y-F(X)\|^2}{\|F(X)-X\|+\|Y-F(X)\|}\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-X\|\right]+\delta\left[\|Y-X\|+\|F(X)-F(Y)\|\right]+\eta\left[\|Y-F(X)\|\right]$$

$$=\alpha\left[\frac{\|Y-F(Y)\|\|Y-F(X)\|+\|F(X)-X\|\|F(X)-F(Y)\|+\|Y-F(X)\|^2}{\|F(X)-F(Y)\|}\right]$$

$$+\beta\left[\frac{\|F(X)-X\|\|Y-F(X)\|+\|Y-F(Y)\|\|Y-X\|+\|Y-F(X)\|^2}{\|Y-X\|}\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-X\|\right]+\delta\left[\|Y-X\|+\|F(X)-F(Y)\|\right]+\eta\left[\|Y-F(X)\|\right]$$

$$=\alpha\left[\frac{\|Y-F(Y)\|\|\tfrac{1}{2}(F+I)(X)-F(X)\|+\|F(X)-X\|\|F(X)-F\left[\tfrac{1}{2}(F+I)(X)\right]\|+\|\tfrac{1}{2}(F+I)(X)-F(X)\|^2}{\|F(X)-F\left[\tfrac{1}{2}(F+I)(X)\right]\|}\right]$$

$$+\beta\left[\frac{\|F(X)-X\|\|\tfrac{1}{2}(F+I)(X)-F(X)\|+\|Y-F(Y)\|\|\left[\tfrac{1}{2}(F+I)(X)-X\right]\|+\|\tfrac{1}{2}(F+I)(X)-F(X)\|^2}{\|\tfrac{1}{2}(F+I)(X)-X\|}\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-X\|\right]+\delta\left[\|\tfrac{1}{2}(F+I)(X)-X\|+\|F(X)-F\left[\tfrac{1}{2}(F+I)(X)\right]\|\right]$$

$$+\eta\|\tfrac{1}{2}(F+I)(X)-F(X)\|$$

$$=\alpha\left[\frac{\|Y-F(Y)\|\tfrac{1}{2}\|F(X)-X\|+\|F(X)-X\|\tfrac{1}{2}\|F(X)-X\|+\tfrac{1}{4}\|F(X)-X\|^2}{\tfrac{1}{2}\|F(X)-X\|}\right]$$

$$+\beta\left[\frac{\|F(X)-X\|\tfrac{1}{2}\|F(X)-X\|+\|Y-F(Y)\|\tfrac{1}{2}\|F(X)-X\|+\tfrac{1}{4}\|F(X)-X\|^2}{\tfrac{1}{2}\|F(X)-X\|}\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-X\|\right]+\delta\left[\tfrac{1}{2}\|F(X)-X\|+\tfrac{1}{2}\|F(X)-X\|\right]+\eta\tfrac{1}{2}\|F(X)-X\|$$

$$=\alpha\left[\|Y-F(Y)\|+\|F(X)-X\|+\tfrac{1}{2}\|F(X)-X\|\right]+\beta\left[\|F(X)-X\|+\|Y-F(Y)\|+\tfrac{1}{2}\|F(X)-X\|\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-X\|\right]+\delta\|F(X)-X\|+\frac{\eta}{2}\|F(X)-X\|$$

$$=\alpha\left[\|Y-F(Y)\|+\tfrac{3}{2}\|F(X)-X\|\right]+\beta\left[\tfrac{3}{2}\|F(X)-X\|+\|Y-F(Y)\|\right]$$

$$+\gamma\left[\|Y-F(Y)\|+\|F(X)-X\|\right]+\delta\|F(X)-X\|+\frac{\eta}{2}\|F(X)-X\|$$

$$=\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|F(X)-X\|+\left[\alpha+\beta+\gamma\right]\|Y-F(Y)\|$$

$$\|Z-X\|\le\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|F(X)-X\|+\left[\alpha+\beta+\gamma\right]\|Y-F(Y)\| \tag{3.1.3}$$

Also

$$\|u-X\|=\|2Y-Z-X\|=\|(F+I)(X)-Z-X\|=\|F(X)+X-Z-X\|$$

$$=\|F(X)-Z\|=\|F(X)-F(Y)\|$$

$$\le\alpha\left[\frac{\|X-F(X)\|\|X-Y\|+\|Y-F(Y)\|\|Y-F(X)\|+\|X-Y\|^2}{\|X-F(X)\|+\|X-Y\|}\right]$$

$$+\beta\left[\frac{\|Y-F(Y)\|\|X-Y\|+\|X-F(X)\|\|X-F(Y)\|+\|X-Y\|^2}{\|Y-F(Y)\|+\|X-Y\|}\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\left[\|X-F(Y)\|+\|Y-F(X)\|\right]+\eta\|X-Y\|$$

$$=\alpha\left[\frac{\|X-F(X)\|\|X-Y\|+\|Y-F(Y)\|\|Y-F(X)\|+\|X-Y\|^2}{\|Y-F(X)\|}\right]$$

$$+\beta\left[\frac{\|Y-F(Y)\|\|X-Y\|+\|X-F(X)\|\|X-F(Y)\|+\|X-Y\|^2}{\|X-F(Y)\|}\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\left[\|X-F(Y)\|+\|Y-F(X)\|\right]+\eta\|X-Y\|$$

$$=\alpha\left[\frac{\|X-F(X)\|\left\|X-\left[\tfrac{1}{2}(F+I)(X)\right]\right\|+\|Y-F(Y)\|\left\|\tfrac{1}{2}(F+I)(X)-F(X)\right\|+\left\|X-\left[\tfrac{1}{2}(F+I)(X)\right]\right\|^2}{\left\|\tfrac{1}{2}(F+I)(X)-F(X)\right\|}\right]$$

$$+\beta\left[\frac{\|Y-F(Y)\|\left\|X-\left[\tfrac{1}{2}(F+I)(X)\right]\right\|+\|X-F(X)\|\left\|X-F\left[\tfrac{1}{2}(F+I)(X)\right]\right\|+\left\|X-\left[\tfrac{1}{2}(F+I)(X)\right]\right\|^2}{\left\|X-F\left[\tfrac{1}{2}(F+I)(X)\right]\right\|}\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\left[\left\|X-F\left[\tfrac{1}{2}(F+I)(X)\right]\right\|+\left\|\tfrac{1}{2}(F+I)(X)-F(X)\right\|\right]$$
$$+\eta\left[\left\|X-\left[\tfrac{1}{2}(F+I)(X)\right]\right\|\right]$$

$$=\alpha\left[\frac{\|X-F(X)\|\tfrac{1}{2}\|X-F(X)\|+\|Y-F(Y)\|\tfrac{1}{2}\|X-F(X)\|+\tfrac{1}{4}\|X-F(X)\|^2}{\tfrac{1}{2}\|X-F(X)\|}\right]$$

$$+\beta\left[\frac{\|Y-F(Y)\|\tfrac{1}{2}\|X-F(X)\|+\|X-F(X)\|\tfrac{1}{2}\|X-F(X)\|+\tfrac{1}{4}\|X-F(X)\|^2}{\tfrac{1}{2}\|X-F(X)\|}\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\left[\tfrac{1}{2}\|X-F(X)\|+\tfrac{1}{2}\|X-F(X)\|\right]+\eta\left[\tfrac{1}{2}\|X-F(X)\|\right]$$

$$=\alpha\left[\|X-F(X)\|+\|Y-F(Y)\|+\tfrac{1}{2}\|X-F(X)\|\right]+\beta\left[\|Y-F(Y)\|+\|X-F(X)\|+\tfrac{1}{2}\|X-F(X)\|\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\left[\|X-F(X)\|\right]+\frac{\eta}{2}\|X-F(X)\|$$

$$=\alpha\left[\tfrac{3}{2}\|X-F(X)\|+\|Y-F(Y)\|\right]+\beta\left[\|Y-F(Y)\|+\tfrac{3}{2}\|X-F(X)\|\right]$$

$$+\gamma\left[\|X-F(X)\|+\|Y-F(Y)\|\right]+\delta\|X-F(X)\|+\frac{\eta}{2}\|X-F(X)\|$$

$$=\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|X-F(X)\|+[\alpha+\beta+\gamma]\|Y-F(Y)\|$$

$$\therefore\|u-X\|\le\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|X-F(X)\|+[\alpha+\beta+\gamma]\|Y-F(Y)\| \qquad (3.1.4)$$

Now,

$$\|Z-u\|\le\|Z-X\|+\|X-u\|$$

$$=\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|X-F(X)\|+[\alpha+\beta+\gamma]\|Y-F(Y)\|$$

$$+\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|X-F(X)\|+[\alpha+\beta+\gamma]\|Y-F(Y)\|$$

$$=2\left[\frac{3}{2}\alpha+\frac{3}{2}\beta+\gamma+\delta+\frac{\eta}{2}\right]\|X-F(X)\|+2[\alpha+\beta+\gamma]\|Y-F(Y)\|$$

$$= [3\alpha + 3\beta + 2\gamma + 2\delta + \eta]\|X - F(X)\| + [2\alpha + 2\beta + 2\gamma]\|Y - F(Y)\|$$

$$\|Z - u\| = [3\alpha + 3\beta + 2\gamma + 2\delta + \eta]\|X - F(X)\| + [2\alpha + 2\beta + 2\gamma]\|Y - F(Y)\| \tag{3.1.5}$$

Also

$$\|Z - u\| = \|F(Y) - (2Y - Z)\|$$

$$= \|F(Y) - 2Y + Z\|$$

$$= 2\|F(Y) - Y\|$$

$\therefore$ from (1.5)

$$\therefore\ 2\|Y - F(Y)\| = [3\alpha + 3\beta + 2\gamma + 2\delta + \eta]\|X - F(X)\| + [2\alpha + 2\beta + 2\gamma]\|Y - F(Y)\|$$

$$\therefore \|Y - F(Y)\| \le q\|X - F(X)\|$$

where $q = \dfrac{3\alpha + 3\beta + 2\gamma + 2\delta + \eta}{2 - (2\alpha + 2\beta + 2\gamma)} < 1$

since $5\alpha + 5\beta + 4\gamma + 2\delta + \eta < 2$

Let $G = \frac{1}{2}(F + I)$ then for every $x \in X$

$$\|G^2(X) - G(X)\| = \|G(Y) - Y\|$$

$$= \left\|\frac{1}{2}(F + I)(Y) - Y\right\|$$

$$= \frac{1}{2}\|Y - F(Y)\|$$

$$< \frac{q}{2}\|X - F(X)\|$$

By the definition of q, we claim that $\{G^n(X)\}$ is a Cauchy sequence in X.

By the completeness, $\{G^n(X)\}$ converges to some element X_0 in X.

i.e. $\lim_{n\to\infty} G^n(X) = X_0$

Which implies that $G(X_0) = X_0$.

Hence $F(X_0) = X_0$

i.e. X_0 is a fixed point of F.

For the uniqueness, if possible let $Y_0 (\ne X_0)$ be another fixed point of F then

$$\|X_0 - Y_0\| = \|F(X_0) - F(Y_0)\|$$

$$\le \alpha\left[\frac{\|X_0 - F(X_0)\|\|X_0 - Y_0\| + \|Y_0 - F(Y_0)\|\|Y_0 - F(X_0)\| + \|X_0 - Y_0\|^2}{\|X_0 - F(X_0)\| + \|X_0 - Y_0\|}\right]$$

$$+\beta\left[\frac{\|Y_0 - F(Y_0)\|\|X_0 - Y_0\| + \|X_0 - F(X_0)\|\|X_0 - F(Y_0)\| + \|X_0 - Y_0\|^2}{\|Y_0 - F(Y_0)\| + \|X_0 - Y_0\|}\right]$$

$$+\gamma\left[\left\|X_0-F(X_0)\right\|+\left\|Y_0-F(Y_0)\right\|\right]+\delta\left[\left\|X_0-F(Y_0)\right\|+\left\|Y_0-F(Y_0)\right\|\right]+\eta\left\|X_0-Y_0\right\|$$

$$=\alpha\frac{\left\|X_0-Y_0\right\|^2}{\left\|X_0-Y_0\right\|}+\beta\frac{\left\|X_0-Y_0\right\|^2}{\left\|X_0-Y_0\right\|}+2\delta\left\|X_0-Y_0\right\|+\eta\left\|X_0-Y_0\right\|$$

$$=\alpha\left\|X_0-Y_0\right\|+\beta\left\|X_0-Y_0\right\|+2\delta\left\|X_0-Y_0\right\|+\eta\left\|X_0-Y_0\right\|$$

$$=\left[\alpha+\beta+2\delta+\eta\right]\left\|X_0-Y_0\right\|$$

$$\therefore \left\|X_0-Y_0\right\|\le\left[\alpha+\beta+2\delta+\eta\right]\left\|X_0-Y_0\right\|$$

since $\alpha+\beta+2\delta+\eta<1$

$$\therefore \left\|X_0-Y_0\right\|=0$$

$$\therefore X_0=Y_0$$

This completes the proof.

Theorem 3.2: Let K be closed and convex subject of a Banach space X. Let F : K → K, G : K → K satisfy the following conditions :

F and *G* commute (3.2.1)

$F^2 = I$ and $G^2 = I$, where *I* denotes identify mapping (3.2.2)

$$\left\|F(X)-F(Y)\right\| \tag{3.2.3}$$

$$\le\alpha\left[\frac{\left\|G(X)-F(X)\right\|\left\|G(X)-G(Y)\right\|+\left\|G(Y)-F(Y)\right\|\left\|G(Y)-F(X)\right\|+\left\|G(X)-G(Y)\right\|^2}{\left\|G(X)-F(X)\right\|+\left\|G(X)-G(Y)\right\|}\right]$$

$$+\beta\left[\frac{\left\|G(Y)-F(Y)\right\|\left\|G(X)-G(Y)\right\|+\left\|G(X)-F(X)\right\|\left\|G(X)-F(Y)\right\|+\left\|G(X)-G(Y)\right\|^2}{\left\|G(Y)-F(Y)\right\|+\left\|G(X)-G(Y)\right\|}\right]$$

$$+\gamma\left[\left\|G(X)-F(X)\right\|+\left\|G(Y)-F(Y)\right\|\right]+\delta\left[\left\|G(X)-F(Y)\right\|+\left\|G(Y)-F(X)\right\|\right]+\eta\left\|G(X)-G(Y)\right\|$$

For every $X,Y,\in X, 0\le\alpha,\beta,\gamma,\delta,\eta$ and $5\alpha+5\beta+4\gamma+2\delta+\eta<2$. Then there exist at least one fixed point, $X_0\in X$ such that $F(X_0)=G(X_0)=X_0$. further if $\alpha+\beta+2\delta+\eta<1$ then *X* is the unique fixed point of *F* and *G*.

Proof:

from (3.2.1) and (3.2.2) it follows that $(FG)^2 = I$ and (3.2.2) and (3.2.3) imply.

$$\left\|FGG(X)-FGG(Y)\right\|=\left\|FG^2(X)-FG^2(Y)\right\|$$

$$\le\alpha\left[\frac{\left\|GG^2(X)-FG^2(X)\right\|\left\|GG^2(X)-GG^2(Y)\right\|+\left\|GG^2(Y)-FG^2(Y)\right\|\left\|GG^2(Y)-FG^2(X)\right\|+\left\|GG^2(X)-GG^2(Y)\right\|^2}{\left\|GG^2(X)-FG^2(X)\right\|+\left\|GG^2(X)-GG^2(Y)\right\|}\right]$$

$$+\beta\left[\frac{\left\|GG^2(Y)-FG^2(Y)\right\|\left\|GG^2(X)-GG^2(Y)\right\|+\left\|GG^2(X)-FG^2(X)\right\|\left\|GG^2(X)-FG^2(Y)\right\|+\left\|GG^2(X)-GG^2(Y)\right\|^2}{\left\|GG^2(Y)-FG^2(Y)\right\|+\left\|GG^2(X)-GG^2(Y)\right\|}\right]$$

$$+\gamma\left[\left\|GG^2(X)-FG^2(X)\right\|+\left\|GG^2(Y)-FG^2(Y)\right\|\right]$$

$$+\delta\left[\left\|GG^2(X)-FG^2(Y)\right\|+\left\|GG^2(Y)-FG^2(X)\right\|\right]+\eta\left\|GG^2(X)-GG^2(Y)\right\|$$

$$\leq\alpha\left[\frac{\|G(X)-FG\cdot G(X)\|\|G(X)-G(Y)\|+\|G(Y)-FG\cdot G(Y)\|\|G(Y)-FG\cdot G(X)\|+\|G(X)-G(Y)\|^2}{\|G(X)-FG\cdot G(X)\|+\|G(X)-G(Y)\|}\right]$$

$$+\beta\left[\frac{\|G(Y)-FG\cdot G(Y)\|\|G(X)-G(Y)\|+\|G(X)-FG\cdot G(X)\|\|G(X)-FG\cdot G(Y)\|+\|G(X)-G(Y)\|^2}{\|G(Y)-FG\cdot G(Y)\|+\|G(X)-G(Y)\|}\right]$$

$$+\gamma\left[\|G(X)-FG\cdot G(X)\|+\|G(Y)-FG\cdot G(Y)\|\right]$$

$$+\delta\left[\|G(X)-FG\cdot G(Y)\|+\|G(Y)-FG\cdot G(X)\|\right]+\eta\left[\|G(X)-G(Y)\|\right]$$

Now that $G(X)=Z$ and $G(Y)=W$, then we get

$$\|FG(Z)-FG(W)\|$$

$$\leq\alpha\left[\frac{\|Z-FG(Z)\|\|Z-W\|+\|W-FG(W)\|\|W-FG(Z)\|+\|Z-W\|^2}{\|Z-FG(Z)\|+\|Z-W\|}\right]$$

$$+\beta\left[\frac{\|W-FG(W)\|\|Z-W\|+\|Z-FG(Z)\|\|Z-FG(W)\|+\|Z-W\|^2}{\|W-FG(W)\|+\|Z-W\|}\right]$$

$$+\gamma\left[\|Z-FG(Z)\|+\|W-FG(W)\|\right]+\delta\left[\|Z-FG(W)\|+\|W-FG(Z)\|\right]+\eta\|Z-W\|$$

We have $(FG)^2=I$ and so by theorem I, FG has at least one fixed point say X_0 in K, i.e.

$$FG(X_0)=X_0 \tag{3.2.4}$$

$$FFG(X_0)=F(X_0)$$

$$G(X_0)=F(X_0)$$

Now,

$$\|F(X_0)-X_0\|=\|F(X_0)-F^2(X_0)\|=\|F(X_0)-FF(X_0)\|$$

$$\leq\alpha\left[\frac{\|G(X_0)-F(X_0)\|\|G(X_0)-GF(X_0)\|+\|GF(X_0)-FF(X_0)\|\|GF(X_0)-F(X_0)\|+\|G(X_0)-GF(X_0)\|^2}{\|G(X_0)-F(X_0)\|+\|G(X_0)-GF(X_0)\|}\right]$$

$$+\beta\left[\frac{\|GF(X_0)-FF(X_0)\|\|G(X_0)-GF(X_0)\|+\|G(X_0)-F(X_0)\|\|G(X_0)-FF(X_0)\|+\|G(X_0)-GF(X_0)\|^2}{\|GF(X_0)-FF(X_0)\|+\|G(X_0)-GF(X_0)\|}\right]$$

$$+\gamma\left[\|G(X_0)-F(X_0)\|+\|GF(X_0)-FF(X_0)\|\right]$$

$$+\delta\left[\|G(X_0)-FF(X_0)\|+\|GF(X_0)-F(X_0)\|\right]+\eta\|G(X_0)-GF(X_0)\|$$

$$=\alpha\left[\frac{\|F(X_0)-F(X_0)\|\|F(X_0)-X_0\|+\|X_0-X_0\|\|X_0-F(X_0)\|+\|F(X_0)-X_0\|^2}{\|F(X_0)-F(X_0)\|+\|F(X_0)-X_0\|}\right]$$

$$+\beta\left[\frac{\|X_0-X_0\|\|F(X_0)-X_0\|+\|F(X_0)-F(X_0)\|\|F(X_0)-X_0\|+\|F(X_0)-X_0\|^2}{\|X_0-X_0\|+\|F(X_0)-X_0\|}\right]$$

$$+\gamma\left[\left\|F(X_0)-F(X_0)\right\|+\left\|X_0-X_0\right\|\right]+\delta\left[\left\|F(X_0)-X_0\right\|+\left\|X_0-F(X_0)\right\|\right]+\eta\left[\left\|F(X_0)-(X_0)\right\|\right]$$

$$=\alpha\left\|F(X_0)-X_0\right\|+\beta\left\|F(X_0)-X_0\right\|+2\delta\left\|F(X_0)-X_0\right\|+\eta\left\|F(X_0)-X_0\right\|$$

$$=(\alpha+\beta+2\delta+\eta)\left\|F(X_0)-X_0\right\|$$

There fore

$$\left\|F(X_0)-X_0\right\|\leq(\alpha+\beta+2\delta+\eta)\left\|F(X_0)-X_0\right\|$$

This is contradiction

Since $\alpha+\beta+\delta+\eta<1$

$\therefore F(X_0)=X_0$

I.e. X_0 is fixed point of *F*, but $F(X_0)=G(X_0)$ therefore we have $G(X_0)=X_0$

I.e. X_0 is the common fixed point of *F* and *G*.

Now, we shall prove that X_0 is the unique common fixed point of *F* and *G*. If possible let Y_0 be another fixed point of *F* and *G*.

Now by (3.2.1), (3.2.2), (3.2.3), (3.2.4) and (3.2.5) we have

$$\left\|X_0-Y_0\right\|=\left\|F^2(X_0)-F^2(Y_0)\right\|=\left\|FF(X_0)-FF(Y_0)\right\|$$

$$\leq\alpha\left[\frac{\left\|GF(X_0)-FF(X_0)\right\|\left\|GF(X_0)-GF(Y_0)\right\|+\left\|GF(Y_0)-FF(Y_0)\right\|\left\|GF(Y_0)-FF(X_0)\right\|+\left\|GF(X_0)-GF(Y_0)\right\|^2}{\left\|GF(X_0)-FF(X_0)\right\|+\left\|GF(X_0)-GF(Y_0)\right\|}\right]$$

$$+\beta\left[\frac{\left\|GF(Y_0)-FF(Y_0)\right\|\left\|GF(X_0)-GF(Y_0)\right\|+\left\|GF(X_0)-FF(X_0)\right\|\left\|GF(X_0)-FF(Y_0)\right\|+\left\|GF(X_0)-GF(Y_0)\right\|^2}{\left\|GF(Y_0)-FF(Y_0)\right\|+\left\|GF(X_0)-GF(Y_0)\right\|}\right]$$

$$+\gamma\left[\left\|GF(X_0)-FF(X_0)\right\|+\left\|GF(Y_0)-FF(Y_0)\right\|\right]$$
$$+\delta\left[\left\|GF(X_0)-FF(Y_0)\right\|+\left\|GF(Y_0)-FF(X_0)\right\|\right]+\eta\left\|GF(X_0)-GF(Y_0)\right\|$$

$$=\alpha\left\|X_0-Y_0\right\|+\beta\left\|X_0-Y_0\right\|+2\delta\left\|X_0-Y_0\right\|+\eta\left\|X_0-Y_0\right\|$$

$$=(\alpha+\beta+2\delta+\eta)\left\|X_0-Y_0\right\|$$

$$\left\|X_0-Y_0\right\|\leq(\alpha+\beta+2\delta+\eta)\left\|X_0-Y_0\right\|$$

Since $\alpha+\beta+2\delta+\eta<1,$ it follows

$$X_0=Y_0$$

Proving the uniqueness of X_0, the proof of theorem 2 is complete.

Theorem 3.3:

Let K be a closed and convex subset of a Banach space X. Let F, G and H be three mappings of X into itself such that

$FG=GF,\ GH=HG$ and $FH=HF$ (3.3.1)

$F^2=I,\ G^2=I,\ H^2=I,$ where I denotes the identify mapping (3.3.2)

$\left\|F(X)-F(Y)\right\|$ (3.3.3)

$$\leq \alpha\left[\frac{\|GH(X)-F(X)\|\|GH(X)-GH(Y)\|+\|GH(Y)-F(Y)\|\|GH(Y)-F(X)\|+\|GH(X)-GH(Y)\|^2}{\|GH(X)-F(X)\|+\|GH(X)-GH(Y)\|}\right]$$

$$+\beta\left[\frac{\|GH(Y)-F(Y)\|\|GH(X)-GH(Y)\|+\|GH(X)-F(X)\|\|GH(X)-F(Y)\|+\|GH(X)-GH(Y)\|^2}{\|GH(Y)-F(Y)\|+\|GH(X)-GH(Y)\|}\right]$$

$$+\gamma\left[\|GH(X)-F(X)\|+\|GH(Y)-F(Y)\|\right]+\delta\left[\|GH(X)-F(Y)\|+\|GH(Y)-F(X)\|\right]$$
$$+\eta\|GH(X)-GH(Y)\|$$

For every $X,Y\in K$ and $0\leq \alpha,\ \beta,\ \gamma,\ \delta,\ \eta$ such that $5\alpha+5\beta+4\gamma+2\delta+\eta<2$ then there exist at least one fixed point $X_0\in X$ such that

$$F(X_0)=GH(X_0) \text{ and } FG(X_0)=H(X_0)$$

Further if $\alpha+\beta+2\delta+\eta<1$ then F has a unique fixed point.

Proof: From (3.3.1) and (3.3.2) it follows that $(FGH)^2=I,$ where I is the identify mapping, from (3.3.2) and (3.3.3) we have

$$\|FGH\cdot G(X)-FGH\cdot G(Y)\|=\|F\cdot GHG(X)-F\cdot GHG(Y)\|$$

$$\leq \alpha\left[\frac{\|(GH)^2G(X)-FGHG(X)\|\|(GH)^2G(X)-(GH)^2G(Y)\|+\|(GH)^2G(Y)-FGHG(Y)\|\|(GH)^2G(Y)-FGHG(X)\|+\|(GH)^2G(X)-(GH)^2G(Y)\|^2}{\|(GH)^2G(X)-FGHG(X)\|+\|(GH)^2G(X)-(GH)^2G(Y)\|}\right]$$

$$+\beta\left[\frac{\|(GH)^2G(Y)-FGHG(Y)\|\|(GH)^2G(X)-(GH)^2G(Y)\|+\|(GH)^2G(X)-FGHG(X)\|\|(GH)^2G(X)-FGHG(Y)\|+\|(GH)^2G(X)-(GH)^2G(Y)\|^2}{\|(GH)^2G(Y)-FGHG(Y)\|+\|(GH)^2G(X)-(GH)^2G(Y)\|}\right]$$

$$+\gamma\left[\|(GH)^2G(X)-FGHG(X)\|+\|(GH)^2G(Y)-FGHG(Y)\|\right]$$
$$+\delta\left[\|(GH)^2G(X)-FGHG(Y)\|+\|(GH)^2G(Y)-FGHG(X)\|\right]$$
$$+\eta\left[\|(GH)^2G(X)-(GH)^2G(Y)\|\right]$$

$$=\alpha\left[\frac{\|G(X)-FGHG(X)\|\|G(X)-G(Y)\|+\|G(Y)-FGHG(Y)\|\|G(Y)-FGHG(X)\|+\|G(X)-G(Y)\|^2}{\|G(X)-FGHG(X)\|+\|G(X)-G(Y)\|}\right]$$

$$+\beta\left[\frac{\|G(Y)-FGHG(Y)\|\|G(X)-G(Y)\|+\|G(X)-FGHG(X)\|\|G(X)-FGHG(Y)\|+\|G(X)-G(Y)\|^2}{\|G(Y)-FGHG(Y)\|+\|G(X)-G(Y)\|}\right]$$

$$+\gamma\left[\|G(X)-FGHG(X)\|+\|G(Y)-FGHG(Y)\|\right]+\delta\left[\|G(X)-FGHG(Y)\|+\|G(Y)-FGHG(X)\|\right]$$
$$+\eta\left[\|G(X)-G(Y)\|\right]$$

Now, if we put $G(X)=Z$ and $G(Y)=W,$ we get,

$$\|FGH(Z)-FGH(W)\|$$

$$\leq \alpha\left[\frac{\|Z-FGH(Z)\|\|Z-W\|+\|W-FGH(W)\|\|W-FGH(Z)\|+\|Z-W\|^2}{\|Z-FGH(Z)\|+\|Z-W\|}\right]$$

$$+\beta\left[\frac{\|W-FGH(W)\|\|Z-W\|+\|Z-FGH(Z)\|\|Z-FGH(W)\|+\|Z-W\|^2}{\|W-FGH(W)\|+\|Z-W\|}\right]$$
$$+\gamma\left[\|Z-FGH(Z)\|+\|W-FGH(W)\|\right]+\delta\left[\|Z-FGH(W)\|+\|W-FGH(Z)\|\right]$$
$$+\eta\left[\|Z-W\|\right]$$

Put $FGH = N$ then

$$\|N(z)-N(w)\|$$
$$\leq\alpha\left[\frac{\|z-N(z)\|\|z-w\|+\|w-N(w)\|\|w-N(z)\|+\|z-w\|^2}{\|z-N(z)\|+\|z-w\|}\right]$$
$$+\beta\left[\frac{\|w-N(w)\|\|z-w\|+\|z-N(z)\|\|z-N(w)\|+\|z-w\|^2}{\|w-N(w)\|+\|z-w\|}\right]$$
$$+\gamma\left[\|z-N(z)\|+\|w-N(w)\|\right]+\delta\left[\|z-N(w)\|+\|w-N(z)\|\right]+\eta\|z-w\|$$

Put $w=\frac{1}{2}(N+I)(z)$

$N(w)=s$ and $t=2w-s$ (A)

Now from (A) we have

$$\|s-z\|=\|N(w)-z\|=\|N(w)-N^2(z)\|=\|N(w)-N(N(z))\|$$
$$\leq\alpha\left[\frac{\|w-N(w)\|\|w-N(z)\|+\|N(z)-N^2(z)\|\|N(z)-N(w)\|+\|w-N(z)\|^2}{\|w-N(w)\|+\|w-N(z)\|}\right]$$
$$+\beta\left[\frac{\|N(z)-N^2(z)\|\|w-N(z)\|+\|w-N(w)\|\|w-N^2(z)\|+\|w-N(z)\|^2}{\|N(z)-N^2(z)\|+\|w-N(z)\|}\right]$$
$$+\gamma\left[\|w-N(w)\|+\|N(z)-N^2(z)\|\right]+\delta\left[\|w-N^2(z)\|+\|N(z)-N(w)\|\right]+\eta\left[\|w-N(z)\|\right]$$

$$\leq\alpha\left[\frac{\|w-N(w)\|\|w-N(z)\|+\|N-(z)-z\|\|N(z)-N(w)\|+\|w-N(z)\|^2}{\|N(z)-N(w)\|}\right]$$
$$+\beta\left[\frac{\|N(z)-z\|\|w-N(z)\|+\|w-N(w)\|\|w-z\|+\|w-N(z)\|^2}{\|z-w\|}\right]$$
$$+\gamma\left[\|w-N(w)\|+\|N(z)-z\|\right]+\delta\left[\|w-z\|+\|N(z)-N(w)\|\right]+\eta\|w-N(z)\|$$

$$\leq\alpha\left[\frac{\|w-N(w)\|\|\frac{1}{2}(N+I)(z)-N(z)\|+\|N(z)-z\|\|N(z)-N(\frac{1}{2}(N+I)(z))\|+\|\frac{1}{2}(N+I)(z)-N(z)\|^2}{\|N(z)-N(\frac{1}{2}(N+I)(z))\|}\right]$$

$$+\beta\left[\frac{\|N(z)-z\|\|\frac{1}{2}(N+I)(z)-N(z)\|+\|w-N(w)\|\|\frac{1}{2}(N+I)(z)-z\|+\|\frac{1}{2}(N+I)(z)-N(z)\|^2}{\|z-\frac{1}{2}(N+I)(z)\|}\right]$$
$$+\gamma\left[\|w-N(w)\|+\|N(z)-z\|\right]+\delta\left[\|\tfrac{1}{2}(N+I)(z)-z\|+\|N(z)-N(\tfrac{1}{2}(N+I)(z))\|\right]$$

$$+\eta\left\|\tfrac{1}{2}(N+I)(z)-N(z)\right\|$$

$$\leq \alpha\left[\frac{\|w-N(w)\|\frac{1}{2}\|N(z)-z\|+\|N(z)-z\|\frac{1}{2}\|N(z)-z\|+\frac{1}{4}\|N(z)-z\|^2}{\frac{1}{2}\|N(z)-z\|}\right]$$

$$+\beta\left[\frac{\|N(z)-z\|\frac{1}{2}\|N(z)-z\|+\|w-N(w)\|\frac{1}{2}\|N(z)-z\|+\frac{1}{4}\|N(z)-z\|^2}{\frac{1}{2}\|N(z)-z\|}\right]$$

$$+\gamma\left[\|w-N(w)\|+\|N(z)-z\|\right]+\delta\left[\tfrac{1}{2}\|N(z)-z\|+\tfrac{1}{2}\|N(z)-z\|\right]+\eta\left[\tfrac{1}{2}\|N(z)-z\|\right]$$

$$\leq \alpha\left[\|w-N(w)\|+\|N(z)-z\|+\tfrac{1}{2}\|N(z)-z\|\right]$$
$$+\beta\left[\|N(z)-z\|+\|w-N(w)\|+\tfrac{1}{2}\|N(z)-z\|\right]$$
$$+\gamma\left[\|w-N(w)\|+\|N(z)-z\|\right]+\delta\left[\|N(z)-z\|\right]+\eta\tfrac{1}{2}\|N(z)-z\|$$
$$\leq\left(\tfrac{3}{2}\alpha+\tfrac{3}{2}\beta+\gamma+\delta+\tfrac{1}{2}\eta\right)\|N(z)-z\|+(\alpha+\beta+\gamma)\|w-N(w)\| \qquad \text{(B)}$$

Similarly it can be shown that

$$\|t-z\|\leq\left(\tfrac{3}{2}\alpha+\tfrac{3}{2}\beta+\gamma+\delta+\tfrac{1}{2}\eta\right)\|N(z)-z\|+(\alpha+\beta+\gamma)\|w-N(w)\| \qquad \text{(C)}$$

Now

$$\|s-t\|\leq\|s-z\|+\|z-t\|$$
$$\leq(3\alpha+3\beta+2\gamma+2\delta+\eta)\|N(z)-z\|+2(\alpha+\beta+\gamma)\|w-N(w)\| \qquad \text{(D)}$$

Also

$$\|s-t\|=\|N(w)-(2v-s)\|$$
$$=\|N(w)-2v+N(w)\|$$
$$=2\|N(w)-w\|$$

Putting the above value in equality (D), we have

$$2\|N(w)-w\|\leq(3\alpha+3\beta+2\gamma+2\delta+\eta)\|N(z)-z\|+2(\alpha+\beta+\gamma)\|w-N(w)\|$$
$$\therefore \|N(w)-w\|\leq q\|N(z)-z\|$$

where $q=\dfrac{3\alpha+3\beta+2\gamma+2\delta+\eta}{2-(2\alpha+2\beta+2\gamma)}<1$

since $5\alpha+5\beta+4\gamma+2\delta+\eta<2$

i.e. $\|N(w)-w\|\leq q\|N(z)-z\|$ (E)

Put $G=\frac{1}{2}(N+I)$ then for $z\in X$,

$$\|G^2(z)-G(z)\|=\|G(w)-w\|$$
$$=\left\|\tfrac{1}{2}(N+I)(w)-w\right\|$$
$$=\tfrac{1}{2}\|N(w)-w\|$$

$$\leq \tfrac{q}{2}\left\|N(w)-w\right\|$$

By the definition of q, we claim that $\{G^n(X)\}$ is a Cauchy sequence in X.

By the completeness $\{G^n(X)\}$ convergent to some point X_0 in X.

$$\therefore \lim_{n\to\infty} G^n(X) = X_0$$

which implies that $G(X_0) = X_0$

Hence $N(X_0) = X_0$

$$\therefore FGH(X_0) = X_0 \text{ because } N = FGH \tag{3.3.4}$$

and so

$$GH(FGH)(X_0) = GH(X_0)$$

$$\therefore F(X_0) = GH(X_0) \tag{3.3.5}$$

Also

$$H(FGH)(X_0) = H(X_0)$$

$$\therefore FG(X_0) = H(X_0)$$

Now, by (3.3.1), (3.3.2), (3.3.3), (3.3.4) and (3.3.5) we have

$$\left\|H(X_0)-X_0\right\| = \left\|FG(X_0)-F^2(X_0)\right\|$$

$$= \left\|FG(X_0)-FF(X_0)\right\|$$

$$\leq \alpha\left[\frac{\left\|GHG(X_0)-FG(X_0)\right\|\left\|GHG(X_0)-GHF(X_0)\right\|+\left\|GHF(X_0)-FG(X_0)\right\|\left\|GHF(X_0)-FG(X_0)\right\|+\left\|GHG(X_0)-GHF(X_0)\right\|^2}{\left\|GHG(X_0)-FG(X_0)\right\|+\left\|GHG(X_0)-GHF(X_0)\right\|}\right]$$

$$+\beta\left[\frac{\left\|GHF(X_0)-FF(X_0)\right\|\left\|GHG(X_0)-GHF(X_0)\right\|+\left\|GHG(X_0)-FG(X_0)\right\|\left\|GHG(X_0)-FF(X_0)\right\|+\left\|GHG(X_0)-GHF(X_0)\right\|^2}{\left\|GHF(X_0)-FF(X_0)\right\|+\left\|GHG(X_0)-GHF(X_0)\right\|}\right]$$

$$+\gamma\left[\left\|GHG(X_0)-FG(X_0)\right\|+\left\|GHF(X_0)-FF(X_0)\right\|\right]$$

$$+\delta\left[\left\|GHG(X_0)-FF(X_0)\right\|+\left\|GHF(X_0)-FG(X_0)\right\|\right]+\eta\left[\left\|GHG(X_0)-GHF(X_0)\right\|\right]$$

$$=\alpha\left[\frac{\left\|H(X_0)-H(X_0)\right\|\left\|H(X_0)-X_0\right\|+\left\|X_0-X_0\right\|\left\|X_0-H(X_0)\right\|+\left\|H(X_0)-X_0\right\|^2}{\left\|H(X_0)-H(X_0)\right\|+\left\|H(X_0)-X_0\right\|}\right]$$

$$+\beta\left[\frac{\left\|X_0-X_0\right\|\left\|H(X_0)-(X_0)\right\|+\left\|H(X_0)-H(X_0)\right\|\left\|H(X_0)-X_0\right\|+\left\|H(X_0)-X_0\right\|^2}{\left\|X_0-X_0\right\|+\left\|H(X_0)-(X_0)\right\|}\right]$$

$$+\gamma\left[\left\|H(X_0)-H(X_0)\right\|+\left\|X_0-X_0\right\|\right]+\delta\left[\left\|H(X_0)-X_0\right\|+\left\|X_0-H(X_0)\right\|\right]+\eta\left[\left\|H(X_0)-X_0\right\|\right]$$

$$=\alpha\left\|H(X_0)-X_0\right\|+\beta\left\|H(X_0)-X_0\right\|+2\delta\left\|H(X_0)-X_0\right\|+\eta\left\|H(X_0)-X_0\right\|$$

$$=(\alpha+\beta+2\delta+\eta)\left\|H(X_0)-X_0\right\|$$

There fore $\left\|\mathrm{H}(\mathrm{X}_0)-X_0\right\| \leq (\alpha+\beta+2\delta+\eta)\left\|H(X_0)-X_0\right\|$

This is contradiction

Since $\alpha+\beta+2\delta+\eta<1$.

$\therefore H(X_0) = X_0$

I.e. X_0 is the fixed point of H. Thus we have from (3.2.5)

$G(X_0) = F(X_0)$

Again

$$\|F(X_0) - X_0\| = \|F(X_0) - F^2(X_0)\|$$
$$= \|F(X_0) - FF(X_0)\|$$
$$\le \alpha \left[\frac{\|GH(X_0) - F(X_0)\| \|GH(X_0) - GHF(X_0)\| + \|GHF(X_0) - FF(X_0)\| \|GHF(X_0) - F(X_0)\| + \|GH(X_0) - GHF(X_0)\|^2}{\|GH(X_0) - F(X_0)\| + \|GH(X_0) - GHF(X_0)\|} \right]$$
$$+\beta \left[\frac{\|GHF(X_0) - FF(X_0)\| \|GH(X_0) - GHF(X_0)\| + \|GH(X_0) - F(X_0)\| \|GH(X_0) - FF(X_0)\| + \|GH(X_0) - GHF(X_0)\|^2}{\|GHF(X_0) - FF(X_0)\| + \|GH(X_0) - GHF(X_0)\|} \right]$$
$$+\gamma \left[\|GH(X_0) - F(X_0)\| + \|GHF(X_0) - FF(X_0)\| \right]$$
$$+\delta \left[\|GH(X_0) - FF(X_0)\| + \|GHF(X_0) - F(X_0)\| \right] + \eta \left[\|GH(X_0) - GHF(X_0)\| \right]$$
$$= \alpha \left[\frac{\|F(X_0) - F(X_0)\| \|F(X_0) - X_0\| + \|X_0 - X_0\| \|X_0 - F(X_0)\| + \|F(X_0) - X_0\|^2}{\|F(X_0) - F(X_0)\| + \|F(X_0) - X_0\|} \right]$$
$$+\beta \left[\frac{\|X_0 - X_0\| \|F(X_0) - X_0\| + \|F(X_0) - F(X_0)\| \|F(X_0) - X_0\| + \|F(X_0) - X_0\|^2}{\|X_0 - X_0\| + \|F(X_0) - X_0\|} \right]$$
$$+\gamma \left[\|F(X_0) - F(X_0)\| + \|X_0 - X_0\| \right] + \delta \left[\|F(X_0) - X_0\| + \|X_0 - F(X_0)\| \right] + \eta \left[\|F(X_0) - X_0\| \right]$$
$$= \alpha \|F(X_0) - X_0\| + \beta \|F(X_0) - X_0\| + 2\delta \|F(X_0) - X_0\| + \eta \|F(X_0) - X_0\|$$
$$= (\alpha + \beta + 2\delta + \eta) \|F(X_0) - X_0\|$$

$$\therefore \|F(X_0) - X_0\| \le (\alpha + \beta + 2\delta + \eta) \|F(X_0) - X_0\|$$

Which is contradiction, since $\alpha + \beta + 2\delta + \eta < 1$

$\therefore F(X_0) = X_0$

But $F(X_0) = G(X_0)$

$\therefore F(X_0) = G(X_0) = H(X_0) = X_0$

$\therefore X_0$ is the common fixed point of F, G, H.

Using (3.3.3), (3.3.2), (3.3.3), (3.3.4) and (3.3.5)

$$\|X_0 - Y_0\| = \|F^2(X_0) - F^2(Y_0)\|$$
$$= \|FF(X_0) - FF(Y_0)\|$$
$$\le \alpha \left[\frac{\|GHF(X_0) - FF(X_0)\| \|GHF(X_0) - GHF(Y_0)\| + \|GHF(Y_0) - FF(Y_0)\| \|GHF(Y_0) - FF(X_0)\| + \|GHF(X_0) - GHF(Y_0)\|^2}{\|GHF(X_0) - FF(X_0)\| + \|GHF(X_0) - GHF(Y_0)\|} \right]$$

$$+\beta\left[\frac{\|GHF(Y_0)-FF(Y_0)\|\|GHF(X_0)-GHF(Y_0)\|+\|GHF(X_0)-FF(X_0)\|\|GHF(X_0)-FF(Y_0)\|+\|GHF(X_0)-GHF(Y_0)\|^2}{\|GHF(Y_0)-FF(Y_0)\|+\|GHF(X_0)-GHF(Y_0)\|}\right]$$

$$+\gamma\left[\|GHF(X_0)-FF(X_0)\|+\|GHF(Y_0)-FF(Y_0)\|\right]$$

$$+\delta\left[\|GHF(X_0)-FF(Y_0)\|+\|GHF(Y_0)-FF(X_0)\|\right]+\eta\|GHF(X_0)-GHF(Y_0)\|$$

$$=\alpha\left[\frac{\|X_0-X_0\|\|X_0-Y_0\|+\|Y_0-Y_0\|\|Y_0-X_0\|+\|X_0-Y_0\|^2}{\|X_0-X_0\|+\|X_0-Y_0\|}\right]$$

$$+\beta\left[\frac{\|Y_0-Y_0\|\|X_0-Y_0\|+\|X_0-X_0\|\|X_0-Y_0\|+\|X_0-Y_0\|^2}{\|Y_0-Y_0\|+\|X_0-Y_0\|}\right]$$

$$+\gamma\left[\|X_0-X_0\|+\|Y_0-Y_0\|\right]+\delta\left[\|X_0-Y_0\|+\|Y_0-X_0\|\right]+\eta\|X_0-X_0\|$$

$$=\alpha\|X_0-Y_0\|+\beta\|X_0-Y_0\|+2\delta\|X_0-Y_0\|+\eta\|X_0-Y_0\|$$

$$=(\alpha+\beta+2\delta+\eta)\|X_0-Y_0\|$$

$$\therefore\|X_0-Y_0\|\le(\alpha+\beta+2\delta+\eta)\|X_0-Y_0\|$$

This is contradiction, since $\alpha+\beta+2\delta+\eta<1$

$\therefore X_0=Y_0$.This completes the proof of theorem.

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