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COMMUTATIVITY OF ALTERNATIVE PERIODIC RINGS WITH $[(yx)^m x^m - x^m(xy)^m, x] = 0$

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ABSTRACT

Let R be an alternative periodic ring. In this paper, we prove that if R is an $m(m+1)$-torsion free alternative periodic ring satisfying the properties $[(yx)^m x^m - x^m(xy)^m, x] = 0$ for all x, y in $R\backslash N(R)$ and $N(R)$ is commutative, then R is commutative.

***AMS Mathematics subject classification**: 17.*

***Key words**: Alternative rings, Periodic rings, Center.*

INTRODUCTION

M. A. Khan[4] shown that R is commutative if R satisfies the conditions $[x^m, y^m] = 0$, for all x, y in $R\backslash N(R)$ and $[x,[x,y^m]] = 0$, for all x, y in $R\backslash N(R)$. He also shown that the result is valid if $[x,[x,y^m]] = 0$ is replaced by $[(yx)^m x^m - x^m(xy)^m, x] = 0$, for all x, y in $R\backslash N(R)$. In this paper, we prove that if R is an $m(m+1)$-torsion free alternative periodic ring satisfying the properties $[(yx)^m x^m - x^m(xy)^m, x] = 0$ for all x, y in $R\backslash N(R)$ and $N(R)$ is commutative, then R is commutative.

PRELIMINARIES

Throughout this section R denotes an alternative periodic ring, $Z(R)$ the center of R, $U(R)$ the group of units of R, $J(R)$ the Jacobson radical of R, $N(R)$ the set of all nilpotent elements of R and $C(R)$ the commutator ideal of R.

In order to prove our result, we first state the well-known results:

Lemma: 1[1, Theorem 1] Let R be a periodic ring such that $N(R)$ is commutative. If for each a in $N(R)$ and x in R there exists an integer $m = m(x,a) \geq 1$ such that $[x^m,[x^m,a]] = 0$ and $[x^{m+1},[x^{m+1},a]] = 0$, then R is commutative. In particular, if R is a periodic ring such that $N(R)$ is commutative and $[x,[x,a]] = 0$ for all a in $N(R)$ and x in R, then R is commutative.

Lemma: 2[2, Lemma 4] Let R be a periodic ring and let $f: R \to S$ be a homomorphisim of R onto S, then the nilpotents of S coincide with $f(N(R))$, where $N(R)$ is the set of nilpotents of R.

Lemme: 3[4] Let R satisfy the conditions
(i) $[x^m, y^m] = 0$, for all x, y in $R\backslash N(R)$
(ii) $[(yx)^m x^m - x^m(xy)^m, x] = 0$, for all x, y in $R\backslash N(R)$ and
(iii) for any x, y in R, $m[x,y] = 0$ implies $[x,y] = 0$, then R is commutative.

MAIN RESULT

Thereom: 1 Let $m \geq 1$ be a fixed positive integer and let R be an alternative periodic ring satisfying the properties:
(i) for any x, y in R, $m(m+1)[x,y] = 0$ implies $[x,y] = 0$ and
(ii) $[(yx)^m x^m - x^m(xy)^m, x] = 0$, for all x, y in $R\backslash N(R)$. Further, suppose that $N(R)$ is commutative then R is commutative.

Proof: Since R is periodic and $N(R)$ is commutative, Lemma 3 yields that the commutator ideal $C(R)$ in nil that is $C(R) \subseteq N(R)$ and $N(R)$ forms an ideal of R. But $N(R)$ is commutative and also $(N(R))^2 \subseteq Z(R)$.

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First we show that the idempotents of R are central.

Let $e^2 = e \in R$ and $r \in R$.

By replacing x by e and y by $e + er - ere$ in (ii), we get

$$\big((e + er - ere)e\big)^m e^m - e^m\big((e + er - ere)e\big)^m \in Z(R).$$

This implies that $ere - er \in Z(R)$. Thus $ere - er = e(ere - er) = (ere - er)e = 0$.

So, $ere - er = 0$. Therefore, $ere = er$. Similarly, $ere = re$.

Thus $er = re$, for all r in R.

We now prove the theorem for R with identity 1.

Suppose that a in $N(R)$ and R in $R\backslash N(R)$. By using (ii), we get

$$[b^m(1 + a)^m - (1 + a)^{m+1} b^m (1 + a)^{-1}, 1 + a] = 0, \quad (1)$$

for all a in $N(R)$ and b in $R\backslash N(R)$. This implies that

$\{b^m(1 + a)^m - (1 + a)^{m+1} b^m (1 + a)^{-1}\}(1 + a) = (1 + a)\{b^m(1 + a)^m - (1 + a)^{m+1} b^m (1 + a)^{-1}\}$.

So, $\{b^m(1 + a)^{m+1} - (1 + a)^{m+1} b^m\} = (1 + a)\{b^m(1 + a)^m - (1 + a)^{m+1} b^m (1 + a)^{-1}\}$.

Using the binomial expansion and the condition$((N(R)))^2 \subseteq Z(R)$, we get

$$b^m(1 + (m + 1)a + \cdots . +(m + 1)a^m + a^{m+1}) - (1 + (m + 1)a + \cdots + (m + 1)a^m + a^{m+1})b^m$$
$$= (1 + a)\{b^m(1 + a)^m - (1 + a)^{m+1}\, b(1 + a)^{-1}\},$$

So, $(m + 1)(b^m a - ab^m) = (1 + a)\{b^m(1 + a)^m - (1 + a)^{m+1} b(1 + a)^{-1}\}$. (2)

Since N(R) is a commutative ideal, $(m + 1)(b^m a - ab^m) = b^m a - ab^m$, and also by (2),

We have $(1 + a)(m + 1)(b^m a - ab^m) = (1 + a)\{b^m(1 + a)^m - (1 + a)^{m+1} b(1 + a)^{-1}\}$.

Since a in $N(R)$, $1 + a$ in $U(R)$ and by (1), we get

$$(m + 1)(b^m a - ab^m) = (1 + a)\left\{b^m(1 + a)^m - (1 + a)^{m+1} b\,(1 + a)^{-1}\right\} \in Z(R).$$

This implies that, $(m + 1)[b^m, a] \in Z(R)$. Since R is $m(m + 1)$-torsion free, we get

$[b^m, a] \in Z(R)$, for all a in N(R), b in $R\backslash N(R)$. (3)

Now since $N(R)$ is commutative, (2) implies that

$[b^m, a] \in Z(R)$, for all a in $N(R)$, b in R. (4)

Now let $x_1, x_2, \ldots . x_n \in R$. Then $R\backslash C(R)$ is commutative. So, by Lemma 3, we get

$$(x_1 . x_2 \ldots . x_n)^m - x_1^m . x_2^m \ldots . x_n^m \in C(R) \subseteq N(R).$$

Therefore $N(R)$ is commutative yields that

$[(x_1 \ldots \ldots x_n)^m, a] = [x_1^m \ldots \ldots x_n^m, a]$, for all a in $N(R)$. (5)

Combining (4) and (5), we get

$[x_1^m \ldots \ldots x_n^m, a] \in Z(R)$, for all a in $N(R)$, $x_1, x_2, \ldots . x_n \in R$ and $n \geq 1$. (6)

Let S be the subring generated by the m-th powers of the elements of R. Then by (6), we have

$[x, a] \in Z(S)$, for all a in $N(S)$, x in S. (7)

Here $Z(S)$ and $N(S)$ represent the center of S and the set of nilpotent elements of S respectively. Combining the fact that S is periodic, $N(S)$ is commutative, and (7), Lemma 1 shows that S is commutative. Hence $[x^m, y^m] = 0$, for all x, y in R. This implies that R satisfies $[x^m, y^m] = 0$, for all x, y in R. But R also satisfies (i) and (ii).

By Lemma 3, we get the required result. It follows that for every nonzero idempotent e, eR is commutative and hence $e[x, y] = 0$, for all x, y in R. Thus if a in R is potent with $a^n = a, n > 1$ then $a^{n-1}[a, b] = 0 = [a, b]$, for all b in R. Since every element in a periodic ring is the sum of a potent element and nilpotent element, this gives $N(R) \subseteq Z(R)$ and R is commutative by well-known theorem of Herstein[3].

Example: 1 Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} : \alpha, \beta, \gamma \in GF(3) \right\}$. For $m = 5$, R satisfies all the hypothesis of Theorem 1 except that R is $m(m+1)$-torsion free. But R is not commutative. This shows that the condition m(m+1)-torsion free is essential in Theorem 1.

Example: 2 Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} : \alpha, \beta, \gamma \in GF(3) \right\}$. The ring R satisfies all the hypothesis of Theorem 1 except the hypothesis that N(R) is commutative. This shows that commutativity of N(R) is essential in Theorem 1.

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