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ABOUT SUB-BLOCK MATRIX POLYNOMIAL OF MATRIX RANK

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ABSTRACT

***B**y using the partitioned matrix and the λ −polynomial theory, this paper gives a systematic exposition of sub-block matrix polynomial of matrix rank. And it studies the relationship and transformation between sub-block matrix polynomial of matrix rank and the rank of matrix polynomial. Then, this paper mainly get two conclusions and solve the two guesses of literature 2.*

Keywords： *matrix rank； λ −matrix； primary factor； matrix polynomial rank； guess.*

1. INTRODUCTION

Matrix rank is the important content of the matrix theory. In various kinds of related literature, there are many about the important matrix rank of equality and inequality, but there is seldom about the discussion of the matrix polynomial of the sub-block matrix rank, as well, there is a little about the polynomial matrix rank. In 2004, Li jingchao, Jiang jun and Xiang shibin studied a class of matrix rank identity and its promotion, they got an identity about matrix polynomial rank and put forward two conjectures. By using the portioned matrix and the λ −polynomial theory, this paper gives a systematic exposition of sub – block matrix polynomial of matrix rank and got some better results. In addition, its application would solve the two guesses about the identity of polynomial matrix rank

Guess 1: let $A \in P^{n\times n}$, $k_1, k_2, \cdots k_t \in P$ when $k_1, k_2, \cdots k_t$ meet what conditions, then

$$\sum_{i=1}^{t} rank\left(A + k_i E\right) = (t-1)n + rank\left(\prod_{i=1}^{t}\left(A + k_i E\right)\right).$$

Guess 2: Let $A \in P^{n\times n}$, $k_1, k_2, \cdots k_t \in P$, and $A^2 = E$, When $k_1, k_2, \cdots k_t$ meet some conditions, then

$$\sum_{i=1}^{t} rank\left(A + k_i E\right) = (t-1)n + rank\left(f_1(k_1, k_2, \cdots k_t)A + f_2(k_1, k_2, \cdots k_t)E\right),$$

Where f_1, f_2 are polynomials about $k_1, k_2, \cdots k_t$.

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Lemma 1: let $\mathrm{A} \in F^{m\times n}$, $\mathrm{B} \in F^{p\times q}$, then

$$rank\begin{pmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{pmatrix} = rank(\mathrm{A}) + rank(\mathrm{B}) \ ,$$

Further promote

$$rank\begin{pmatrix} \mathrm{A}_1 & 0 & \cdots & 0 \\ 0 & \mathrm{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{A}_n \end{pmatrix} = rank(\mathrm{A}_1) + \cdots rank(\mathrm{A}_n).$$

Lemma 2: let $\mathrm{B}(\lambda) \in F[\lambda]^{s\times l}$, $C(\lambda) \in F[\lambda]^{l\times t}$, $H(\lambda) = \mathrm{B}(\lambda)\times C(\lambda) \in F[\lambda]^{s\times t}$, then for any matrix $\mathrm{A} \in F^{m\times n}$, $H(\mathrm{A}) = \mathrm{B}(\mathrm{A})\times C(\mathrm{A})$.

Proof: let $\mathrm{B}(\lambda) = (b_{ij}(\lambda))_{s\times l}, C(\lambda) = (c_{ij}(\lambda))_{l\times t}, H(\lambda) = (h_{ij}(\lambda))_{s\times t}$,

Since $H(\lambda) = \mathrm{B}(\lambda)\times C(\lambda)$ then $h_{ij}(\lambda) = \sum_{k=1}^{l} b_{ik}(\lambda)c_{kj}(\lambda)$, then

$h_{ij}(\mathrm{A}) = \sum_{k=1}^{l} b_{ik}(\mathrm{A})c_{kj}(\mathrm{A})$ So $H(\mathrm{A}) = \mathrm{B}(\mathrm{A})\times C(\mathrm{A})$.

Corollary 1: let $\mathrm{B}(\lambda) \in F[\lambda]^{s\times s}$, $\mathrm{A} \in F^{n\times n}$, if $\mathrm{B}(\lambda)$ reversible, then $\mathrm{B}(\mathrm{A})$ is also reversible.

Proof: since $\mathrm{B}(\lambda)$ is reversible then there exits $\mathrm{B}_1(\lambda) \in F(\lambda)^{s\times s}$, such that $\mathrm{B}(\lambda)\mathrm{B}_1(\lambda) = \mathrm{I}_s$, according to lemma 2,

$\mathrm{B}(\mathrm{A})\mathrm{B}_1(\mathrm{A}) = \mathrm{I}_{s\times n}$,So $\mathrm{B}(\mathrm{A})$ reversible.

2. MAIN CONCLUSION

Theorem 1: let $\mathrm{B}(\lambda) \in F[\lambda]^{s\times l}$, $\mathrm{A} \in F^{m\times n}$, then $rank\ (\mathrm{B}(\mathrm{A})) = rank\ (d_1(\mathrm{A})) + \cdots + rank\ (d_r(\mathrm{A}))$,

where $r = rank\ (B(\lambda)), d_k(\lambda)(1 \le k \le r\ )$ are all levels of invariant factor of $\mathrm{B}(\lambda)$.

Proof: $\forall B(\lambda) \in F[\lambda]^{s\times l}, \exists$ reversible $\lambda -$ matrix $P(\lambda) \in F[\lambda]^{s\times s}$, $Q(\lambda) \in F[\lambda]^{l\times l}$ such that

$$B(\lambda) = P(\lambda)\begin{pmatrix} d_1(\lambda) & 0 & \cdots & 0 & 0 \\ 0 & d_2(\lambda) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_r(\lambda) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} Q(\lambda),$$ according to lemma 2

$$B(\mathrm{A})=P(\mathrm{A})\begin{pmatrix} d_1(\mathrm{A}) & 0 & \cdots & 0 & 0 \\ 0 & d_2(\mathrm{A}) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_r(\mathrm{A}) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}Q(\mathrm{A})$$

So, both sides are equal rank, according to corollary 1. $P(\mathrm{A}), Q(\mathrm{A})$ are reversible. Because reversible matrix does not change the rank of matrix,

According to lemma 1 $rank\ (\mathrm{B}(\mathrm{A})) = rank\ (d_1(\mathrm{A})) + \cdots + rank\ (d_r(\mathrm{A}))$.

Theorem 2: let $\mathrm{B}(\lambda) \in F[\lambda]^{s\times l}$, $\mathrm{A} \in F^{n\times n}$, then $rank\ (\mathrm{B}(\mathrm{A})) = rank\ (h_1(\mathrm{A})) + \cdots + rank(h_r(\mathrm{A}))$, where $r = rank(B(\lambda))$; $h_1(\lambda), \cdots, h_r(\lambda) \in F[\lambda]$ are any non-zero polynomial, and irreducible factor product of standard factorization compose all the factor of $\mathrm{B}(\lambda)$.

Proof: the proof is similar to the document 2 345 page proof of theorem 9.

From the above discussion shows that one matrix which block is matrix polynomial, its rank can be converted into the sum of the same number of rank of matrix polynomials. So how to calculate the sum of any set of polynomial matrix rank.

Theorem 3: let $f_1(\lambda), \cdots f_s(\lambda) \in F[\lambda]$ are any non-zero polynomial, $\mathrm{A} \in F^{n\times n}$, then

$rank(f_1(\mathrm{A})) + \cdots rank(f_s(\mathrm{A})) = rank(h_1(\mathrm{A})) + \cdots rank(h_r(\mathrm{A}))$, where: $h_1(\lambda), \cdots, h_s(\lambda) \in F[\lambda]$, and there irreducible factors product of standard factorization is similar to the ($f_1(\lambda), \cdots, f_s(\lambda)$)'s irreducible factors product of standard factorization.

Proof: according to theorem-2, the conclusion is correct.

Corollary 2: if $f(x), g(x)$ relatively coprime, then $rank(f(\mathrm{A})) + rank(g(\mathrm{A})) = \mathrm{n} + rank(f(\mathrm{A})g(\mathrm{A}))$.

Corollary 3: if $f(x), g(x)$ non coprime, $d(x)$ is the largest Convention-style, $m(x)$ is the least common multiple, then $rank(f(\mathrm{A})) + rank(g(\mathrm{A})) = rank(d(\mathrm{A})) + rank(m(\mathrm{A}))$.

Corollary 4: if $f_1(x), f_2(x), \cdots f_m(x)$ prime to each other, then

$rank(f_1(\mathrm{A})) + \cdots + rank(f_m(\mathrm{A})) = (m-1)n + rank(f_1(\mathrm{A}) \times f_2(\mathrm{A}) \cdots f_m(\mathrm{A}))$.

Proof: since $f_1(x), f_2(x), \cdots f_m(x)$ prime to each other, then the ($f_1(x), f_2(x), \cdots f_m(x)$)'s irreducible factors product of standard factorization is equal to the ($1, 1, \cdots, f_1(x) f_2(x) \cdots f_m(x)$)'s irreducible factors product of standard factorization, so $rank(f_1(\mathrm{A})) + \cdots + rank(f_m(\mathrm{A})) = (m-1)n + rank(f_1(\mathrm{A}) \times f_2(\mathrm{A}) \cdots f_m(\mathrm{A}))$.

Corollary 4: solve the two guess of conjectures 1. According to corollary 4, when $k_1, k_2, \cdots k_t$ different from one another, conjecture 1 and conjecture 2 are all established,.

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