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# A NOTE ON SYSTEMS OF SUMMATION INEQUALITIES

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*ABSTRACT*

*In this paper we discuss some systems of summation inequalities. We also discuss the under and over functions of systems of summation equations.*

***Keywords:*** *Difference Equation, Summation Equation, Summation Inequality, Under and Over Function.*

## 1. INTRODUCTION:

Agarwal [1], Kelley and Peterson [9] developed the theory of difference equations and difference inequalities. Some difference inequalities and comparison results are obtained by K. L. Bondar [2, 3]. Some summation and difference inequalities are obtained in K. L. Bondar [4, 5]. K. L. Bondar, V. C. Borkar, S. T. Patil [6, 7] and Dang H., Oppenheimer S.F.[8] obtained the existence and uniqueness results for difference equations. Some differential and integral inequalities are given in [10]. In this paper we discuss about systems of summation inequalities. We also discuss the under and over functions of systems of summation equations.

## 2. PRELIMINARY NOTES

Let $J = \{t_0, t_0 + 1\ldots t_0 + a\}$, $t_0 \geq 0$, $t_0 \in R$, and $E$ be an open subset of $R^n$, consider the difference equations with an initial condition,

$$\Delta u(t) = g(t, u(t)),\ u(t_0) = u_0 \tag{1}$$

where $u_0 \in E$, $u: J \to E$, $g : J \times E \to R^n$.

The function $\phi : J \to R^n$ is said to be a solution of initial value problem (1), if it satisfies

$$\Delta\phi(t) = g(t, \phi(t));\quad \phi(t_0) = u_0.$$

The initial value problem is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

By summation convention $\sum_{s=t_0}^{t_0-1} g(s,u(s)) = 0$ and so $u(t)$ given above is the solution of (1).

## 3. MAIN RESULTS:

**Theorem: 3.1** *Assume that*

(i) *$K : J \times J \times R^n \to R^n$ and $K(t, s, x)$ is nondecreasing in $x$ for each fixed $(t, s)$ and one of the inequalities*

$$x(t) \leq h(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s)), \tag{2}$$

$$y(t) \geq h(t) + \sum_{s=t_0}^{t-1} K(t, s, y(s)) \tag{3}$$

*is strict where $x,y : J \to R^n$ ;*

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(ii) $x(t_0) < y(t_0)$. *Then*

$$x(t) < y(t), \quad t \geq t_0. \tag{4}$$

**Proof:** Assume that the conclusion (4) is false. Then the set

$$Z = \bigcup_{i=1}^{n} [t \in [t_0, \infty) : x_i(t) \geq y_i(t)]$$

is nonempty. Let $t_1 = \inf Z$. By (ii), it is clear $t_1 > t_0$. Furthermore, since $Z$ is closed, $t_1 \in Z$, and consequently there exists an index $j$ such that

$$x_j(t_1) = y_j(t_1),$$

$$x_j(t) < y_j(t), \quad t_0 \leq t < t_1,$$

$$x_i(t) \leq y_i(t), \quad t_0 \leq t < t_1, \quad i \neq j.$$

Since $K$ is monotone nondecreasing in $x$, it follows that

$$K_j(t_1, s, x(s)) \leq K_j(t_1, s, y(s)).$$

Hence, using (2) and (3), we arrive at the inequality

$$\begin{aligned} x_j(t_1) &\leq h_j(t_1) + \sum_{s=t_0}^{t_1-1} K_j(t_1, s, x(s)) \\ &\leq h_j(t_1) + \sum_{s=t_0}^{t_1-1} K_j(t_1, s, y(s)) \\ &< y_j(t_1). \end{aligned}$$

This is a contradiction to the fact that $x_j(t_1) = y_j(t_1)$. Hence $Z$ is empty and the theorem is proved.

Let us now consider the summation operator defined by

$$K\phi = \sum_{s=t_0}^{t_1-1} K \ (t, s, \phi(s)). \tag{5}$$

**Definition: 3.2** *We shall say that the operator K is monotone nondecreasing if, for any* $\phi_1$, $\phi_2$ $: J \to R^n$ *such that, for any* $t_1 > t_0$,

$$\phi_1(t) < \phi_2(t), \quad t_0 \leq t < t_1,$$

*implies*

$$K\phi_1(t_1) \leq K\phi_2(t_1).$$

**Theorem: 3.3** *Let the operator K defined by (5) be monotone nondecreasing. Suppose further that, for* $t > t_0$,

$$x - Kx < y - Ky, \tag{6}$$

*where* $x, y : J \times R^n$. *Then* $x(t_0) < y(t_0)$ *implies*

$$x(t) < y(t), \quad t \geq t_0.$$

**Proof:** Assume that the conclusion of theorem is false. Then set

$$Z = \bigcup_{i=1}^{n} [t \in [t_0, \infty) : x_i(t) \geq y_i(t)]$$

is nonempty. Let $t_1 = \inf Z$. By (ii), it is clear that $t_1 > t_0$. Furthermore, since $Z$ is closed, $t_1 \in Z$, and consequently there exists an index $j$ such that

$$x_j(t_1) = y_j(t_1),$$

$$x_j(t) < y_j(t), \quad t_0 \leq t < t_1,$$

 

$$x_i(t) \leq y_i(t), \quad t_0 \leq t < t_1, \quad i \neq j.$$

Since $K$ is monotone nondecreasing in $x$ and using above inequalities, it follows that,

$$K_j x_j(t_1) \leq K_j y_j(t_1). \tag{7}$$

As a result, (6) and (7) yield

$$\begin{aligned} x_j(t_1) &= x_j(t_1) - K_j x_j(t_1) + K_j x_j(t_1) \\ &< y_j(t_1) - K_j y_j(t_1) + K_j y_j(t_1) \\ &\leq y_j(t_1). \end{aligned}$$

This contradicts the fact that, at $t = t_1$, $x_j(t_1) = y_j(t_1)$, and hence the proof is complete.

**Definition: 3.4** *A function $u : J \to R^n$ is said to be an under function of the system of summation equation*

$$x = j + Kx \tag{8}$$

*if it satisfies the inequality*

$$u < h + Ku.$$

*Similarly u is said to be an over function of (8) if verifies the system of inequality*

$$u > h + Ku,$$

*whereas if u satisfies equation (8), it is said to be a solution of (8).*

**Theorem: 3.5** Let *the operator K defined by (5) be monotone nondecreasing. Suppose that x, y, z: $J \to R^n$ be an under function, a solution and an over function of (8), respectively on J. Then*

$$x(t_0) < y(t_0) < z(t_0)$$

*implies*

$$x(t) < y(t) < z(t), \quad t \geq t_0.$$

**Proof:** As $x(t)$ is an under function and $y(t)$ is a solution of (8) respectively, we have

$$x(t) < h(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s)) \text{ and}$$

$$y(t) = h(t) + \sum_{s=t_0}^{t-1} K(t, s, y(s)).$$

Also if $x(t_0) < y(t_0)$, they by Theorem 3.1, we have

$$x(t) < y(t), \quad t \geq t_0.$$

Similarly using definition of solution, an over function of (8) and by Theorem 3.1 again we obtain

$$y(t) < z(t), \quad t \geq t_0.$$

Hence

$$x(t) < y(t) < z(t), \quad t \geq t_0.$$

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