



γ NEAR-RINGS

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ABSTRACT

In this paper we introduce the concept of γ near-rings. In [8], S. Suryanarayanan and R. Balakrishnan investigated a near-ring N in which every N -subgroup is invariant. Motivated by this concept, we probe into the properties of a near-ring N where every N -subgroup is an ideal. We discuss the properties of this newly introduced structure, obtain a complete characterization and a structure theorem for such near-rings.

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1. INTRODUCTION

Near-rings are generalized rings. If in a ring $(N, +, \cdot)$ with two binary operations '+' and ' \cdot ', we ignore the commutativity of '+' and one of the distributive laws, $(N, +, \cdot)$ becomes a near-ring. If we do not stipulate the left distributive law, $(N, +, \cdot)$ becomes a right near-ring. Throughout this paper, N stands for a right near-ring $(N, +, \cdot)$ with at least two elements. Obviously, $0n = 0$ for all n in N , where '0' denotes the identity of the group $(N, +)$. As in [3], a subgroup $(M, +)$ of $(N, +)$ is called (i) a left N -subgroup of N if $MN \subseteq M$, (ii) an N -subgroup of N if $NM \subseteq M$ and (iii) an invariant N -subgroup of N if M satisfies both (i) and (ii). Again in [3], a normal subgroup $(I, +)$ of $(N, +)$ is called (i) a left ideal if $n(n' + i) - nn' \in I$ for all $n, n' \in N$ and $i \in I$ (ii) a right ideal if $IN \subseteq I$ and (iii) an ideal if I satisfies both (i) and (ii). In [4], N is said to be left bipotent if $Na = Na^2$ for all $a \in N$. In [6], N is called a β_3 near-ring if $xNy = yxN$ for all $x, y \in N$. An ideal I of N is called (i) a prime ideal if for all ideals J, K of N , $JK \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$. (ii) a completely semiprime ideal if for $a \in N$, $a^2 \in I \Rightarrow a \in I$. (iii) an IFP ideal [1], if for $a, b \in N$, $ab \in I \Rightarrow an \in I$ for all n in N . (iv) a semiprime ideal if for all ideals J of N , $J^2 \subseteq I \Rightarrow J \subseteq I$. If $\{0\}$ is a semiprime ideal, then N is called a semiprime near-ring [2.87, p.67 of Pilz [3]]. Also in [3], N is said to have property P_4 if for all ideals I of N , $ab \in I$ implies $ba \in I$ for a, b in N . The concept of a mate function in N has been introduced in [7] with a view to handling the regularity structure with considerable ease. A map ' f ' from N into N is called a mate function for N if $x = xf(x)x$ for all x in N . Also the existence of mate functions is preserved under homomorphisms. By identity 1 of N , we mean only the multiplicative identity of N .

Basic concepts and terms used but left undefined in this paper can be found in Pilz [3].

2. NOTATIONS

- (i) E denotes the set of all idempotents of N (e in N is called an idempotent if $e^2 = e$)
- (ii) L denotes the set of all nilpotents of N (a in N is nilpotent if $a^k = 0$ for some positive integer k)
- (iii) $N_d = \{n \in N / n(x+y) = nx + ny \text{ for all } x, y \text{ in } N\}$ – set of all distributive elements of N .
- (iv) $C(N) = \{n \in N / nx = xn \text{ for all } x \text{ in } N\}$ – centre of N .
- (v) $N_0 = \{n \in N / n0 = 0\}$ – zero-symmetric part of N .

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3. PRELIMINARY RESULTS

We freely make use of the following results and designate them as R(1),R(2), ...etc

R(1) N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N (Problem 14, p.9 of [5])

R(2) If f is a mate function for N, then for every x in N, $xf(x), f(x)x \in E$ and $Nx = Nf(x)x, xN = xf(x)N$ (Lemma 3.2 of [7])

R(3) If $L = \{0\}$ and $N = N_0$ then (i) $xy = 0 \Rightarrow yx = 0$ for all x, y in N (ii) N has Insertion of Factors Property– IFP for short – i.e. for x, y in N, $xy=0 \Rightarrow xny=0$ for all n in N. If N satisfies (i) and (ii) then N is said to have (*, IFP) (Lemma 2.3 of [7])

R(4) N has strong IFP if and only if for all ideals I of N, and for $x, y \in N, xy \in I \Rightarrow xny \in I$ for all $n \in N$ (Proposition 9.2, p.289 of [3])

R(5) N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals is again nonzero (Theorem 1.60, p.25 of [3])

R(6) For any n in N, $(0 : n)$ is a left ideal of N (1.43, p.21 of Pilz [3])

R(7) If N is zero-symmetric, then every left ideal is an N-subgroup (Proposition 1.34(b), p.19 of Pilz [3])

R(8) A zero-symmetric near-ring N has IFP if and only if $(0 : S)$ is an ideal where S is any non-empty subset of N (by 9.3, p.289 of [3])

R(9) A near-ring N is called simple if it has no non-trivial ideals of N. (By 1.36, p.19 of Pilz [3])

R(10) If N is a β_3 near-ring, then every left N-subgroup of N is an N-subgroup of N (Proposition 5.3 (iv) of [6])

4. DEFINITION AND EXAMPLES OF γ NEAR-RINGS

In this section we define γ near-rings and give certain examples of this new concept.

Definition 4.1: We say that a right near-ring N is a γ near-ring if every N-subgroup of N is an ideal of N.

Examples 4.2: (a) The near-ring $(N, +, \cdot)$ defined on Klein's four group $(N,+)$ with $N = \{0, a, b, c\}$ where ' \cdot ' is defined as per scheme 22, p.408 of Pilz [3]

\cdot	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

is a γ near – ring.

(b) Let $(N, +)$ be the Klein's four group as in (a) above. If multiplication is defined as per scheme 11, p.408 of Pilz [3],

\cdot	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

then N is not a γ near-ring, as the N-subgroup $\{0, a\}$ is not an ideal.

5. PROPERTIES OF γ NEAR-RINGS

In this section we prove certain important properties of γ near-rings and give a complete characterization of such near-rings.

Proposition 5.1: Let N be a γ near-ring. If N is a β_3 near-ring with identity and $N = N_d$, then every left N-subgroup is an ideal.

Proof: Since N is a γ near-ring, every N -subgroup of N is an ideal. (1)

Let M be any left N -subgroup of N . Since N is a β_3 near-ring, by R(10), M is an N -subgroup of N . This implies M is an ideal [by (1)].

Therefore, every left N -subgroup of N is an ideal.

Proposition 5.2: Let N be a γ near-ring. Then every left N -subgroup of N is invariant.

Proof: Let M be an N -subgroup of N . Since N is a γ near-ring, M becomes an ideal of N . Now, the desired result follows from the definition of right ideal.

Remark 5.3: The converse of Proposition 5.2 is not valid. For example, consider the near-ring $(N, +, \cdot)$ where $(N, +)$ is the usual group of integers modulo 6 and where ‘ \cdot ’ is defined as per scheme 24,p.408 of Pilz [3]

\cdot	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

We observe that, every N -subgroup of N is invariant. However N is not a γ near-ring, since the N -subgroup $\{0, 3\}$ is not an ideal.

Proposition 5.4: Let N be a γ near-ring which admits a mate function ‘ f ’. Then

- (i) for all N – subgroups A and B of N , $A \cap B = AB$.
- (ii) $Nx \cap Ny = Nxy$ for all x, y in N .

Proof: (i) Let A and B be two N -subgroups of N . Since N is a γ near-ring, A and B are ideals of N . Hence $AN \subseteq A$ and $BN \subseteq B$.

Now, for $x \in A$ and $y \in B$, $xy \in AN \subseteq A$. Therefore, $AB \subseteq A$.

Also, $xy \in NB \subseteq B$. Hence $AB \subseteq B$. Consequently,

$$AB \subseteq A \cap B \tag{1}$$

On the other hand, if $z \in A \cap B$, then since ‘ f ’ is a mate function for N , $z = zf(z)z \in (AN)B \subseteq AB$. Thus

$$A \cap B \subseteq AB \tag{2}$$

Combining (1) and (2) $A \cap B = AB$ for all N -subgroups A, B of N .

(ii) Let $x, y \in N$. Then by taking $A = Nx$ and $B = Ny$ in (i) we get,

$$Nx \cap Ny = NxNy \tag{3}$$

Again by taking $A = Nx$ and $B = N$ in (i) we get, $Nx = Nx \cap N = NxN$. Therefore,

$$Nxy = NxNy \tag{4}$$

From (3) and (4), we get $Nx \cap Ny = Nxy$ for all x, y in N .

We furnish below a characterization theorem for γ near-rings.

Theorem 5.5: Let N be a near-ring which admits a mate function ‘ f ’. Then the following are equivalent.

- (i) N is a γ near-ring.
- (ii) Every N -subgroup is a completely semiprime ideal of N .
- (iii) Every N -subgroup is an IFP ideal.

Proof:

(i) \Rightarrow (ii): Let M be any N-subgroup of N. Since N is a γ near-ring, M becomes an ideal of N. Let $x^2 \in M$. Now, since 'f' is a mate function for N, $x = xf(x)x \in Nx = Nx \cap Nx = Nx^2$ [by Proposition 5.4 (ii)] $\subseteq NM \subseteq M$. Therefore, $x \in M$ and (ii) follows.

(ii) \Rightarrow (iii): Let M be any N-subgroup of N and let $xy \in M$. Now, $(yx)(yx) = y(xy)x \in NMN \subseteq M$ by (ii). Thus we have $(yx)^2 \in M$ and (ii) implies $(yx) \in M$. For all n in N, $(xny)^2 = (xny)(xny) = xn(yx)ny \in NMN \subseteq M$ and again (ii) guarantees that $xny \in M$ and (iii) follows.

(iii) \Rightarrow (i) Obvious.

Proposition 5.6: Let N be a γ near-ring and let N admit a mate function 'f'. Then we have

- (i) N is left bipotent.
- (ii) N has property P₄.
- (iii) N has strong IFP.
- (iv) N is a semiprime near-ring.

Proof: Let N be a γ near-ring and let M be an N-subgroup of N. Then M is an ideal of N.

(i) Since N admits a mate function 'f', we have by Proposition 5.4 (ii), $Nx = Nx \cap Nx = Nx^2$. It follows that $Nx = Nx^2$ and hence (i) follows.

(ii) Let I be an ideal of N. Let $xy \in I$. Now, $(yx)^2 = (yx)(yx) = y(xy)x \in NIN \subseteq IN \subseteq I$. Therefore, $(yx)^2 \in I$. This implies $yx \in I$ [by Proposition 5.5 (ii)]. Consequently, N has property P₄.

(iii) Let $xy \in I$. Then $yx \in I$ [by (ii)]. Now, $yxn \in IN \subseteq I$ for all n in N. This implies $y(xn) \in I$. Therefore, $xny \in I$ [by (ii)] and (iii) follows.

(iv) Let I be any ideal of N such that $I^2 \subseteq M$. Now, for $x \in I$, since 'f' is a mate function for N, $x = xf(x)x \in INI \subseteq I^2 \subseteq M$. Hence $I \subseteq M$. Thus I is a semiprime ideal. In particular, $\{0\}$ is a semiprime ideal of N. Therefore, N is a semiprime near – ring.

We furnish below another characterization of γ near-rings.

Theorem 5.7: Let N admit a mate function 'f' and let $E \subseteq C(N)$. Then N is a γ near-ring if and only if $xN = xNx = Nx^2$ for all x in N.

Proof: Since $E \subseteq C(N)$ we first observe that N is zero-symmetric .

For the 'only if' part, we see that for every x in N, as N is a γ near-ring, Nx, being an N-subgroup, is an ideal of N.

Therefore,

$$(Nx)N \subseteq Nx \tag{1}$$

and

$$N(Nx) \subseteq Nx \tag{2}$$

Hence for any n in N, since 'f' is a mate function for N, $xn = (xf(x)x)n = x(f(x)xn) = xn'x$ for some n' in N [by(1)].

Therefore $xn \in xNx$. Thus $xN \subseteq xNx$. Obviously $xNx \subseteq Nx$ holds. Consequently we have, $xN = xNx$ for all x in N. Again, for any n in N, $nx^2 = n(xf(x)x)x = x(f(x)nx)x$ [since $E \subseteq C(N)$] = $x(n''x)x$ for some n'' in N [by (2)] $\in xNx$. Thus

$$Nx^2 \subseteq xNx \tag{3}$$

For the reverse inclusion, we have for any n in N, $xnx = xf(x)xnx = xn(f(x)xx)$ [since $E \subseteq C(N)$] $\in xNxx$ [by (2)] = $xf(x)Nxx$ [by R(2)] = $Nxf(x)xx$ [since $E \subseteq C(N)$] = Nxx [since 'f' is a mate function for N] = Nx^2 . Therefore, $xnx \in Nx^2$.

Consequently,

$$xNx \subseteq Nx^2 \tag{4}$$

$$\text{Combining (3) and (4), } xNx = Nx^2 \text{ for all x in N} \tag{5}$$

Collecting all these pieces we get, $xN = xNx = Nx^2$ for all x in N

For the 'if' part, first let us show that N has $(*, IFP)$. For any x in N , since ' f ' is a mate function for N , $x = xf(x)x \in xNx = Nx^2$ [by assumption (5)]. Therefore $x = n_1x^2$ for some n_1 in N . This yields that $x^2 = 0 \Rightarrow x = 0$ and $R(1)$ guarantees $L = \{0\}$. Also, since $N = N_0$, from $R(3)$, we see that N has $(*, IFP)$. (6)

We have, by $R(2)$, for any $x \in N$,

$$Nf(x)x = Nx \tag{7}$$

Let $S = \{n - ne/n \in N\}$. We claim that $(0: S) = Nx$. Since $(n - ne)e = 0$ for all n in N , we get $(n - ne)Ne = \{0\}$ [since N has $(*, IFP)$].

Taking $e = f(x)x$, we get, $(n - ne)Nx = \{0\}$ [by (7)]. Consequently,

$$Nx \subseteq (0: S) \tag{8}$$

To prove the reverse inclusion, we consider an arbitrary y in $(0: S)$. Therefore $yS = \{0\}$. Since $f(x)x \in E$, we have $y(y - yf(x)x) = 0 \Rightarrow yf(x)x(y - yf(x)x) = 0$ [since N has $(*, IFP)$] and $\{y(y - yf(x)x)\} - \{yf(x)x(y - yf(x)x)\} = 0$.

Hence $(y - yf(x)x)^2 = 0$. Since $L = \{0\}$, $R(1)$ guarantees that $y - yf(x)x = 0$. Thus $y = yf(x)x \in Nf(x)x = Nx$ [by $R(2)$].

Therefore $y \in Nx$. This implies

$$(0: S) \subseteq Nx \tag{9}$$

From (8) and (9), we get, $Nx = (0: S)$ for all x in N .

Now, $R(8)$ guarantees that Nx is an ideal. If M is any N -subgroup of N , then we have $M = \sum_{x \in M} Nx$. It follows that M is an ideal and hence N becomes a γ near-ring.

Theorem 5.8 Let N be a γ near-ring which admits a mate function ' f ' and let $E \subseteq C(N)$. Then

- (i) any prime ideal of N is a maximal ideal.
- (ii) every N -subgroup of N is a γ near-ring in its own right.

Proof: Let N be a γ near-ring. Since $E \subseteq C(N)$, N is zero-symmetric. Further N has $(*, IFP)$. [by (6) of Theorem 5.7]

(i) Let P be a prime ideal of N . Let J be an ideal of N such that $J \neq P$ and that $P \subset J \subset N$. Let $x \in J - P$. For x in N , since ' f ' is a mate function for N , $x = xf(x)x = f(x)xx$ [since $E \subseteq C(N)$]. Thus for all n in N , $nx = nf(x)x^2$ and this implies $(n - nf(x)x)x = 0$. Since N has $(*, IFP)$, we get $(n - nf(x)x)zx = 0$. And $z(n - nf(x)x)zx = z \cdot 0 = 0$ [since $N = N_0$] for all $z \in N$. Consequently, $N(n - nf(x)x)Nx = \{0\}$. If we let $y = n - nf(x)x$, then $NyNx = \{0\} \subseteq P$. Also, since N is a γ near-ring, Nx, Ny are ideals in N . Since P is prime, we get $Ny \subseteq P$ or $Nx \subseteq P$. If $Nx \subseteq P$ then $x = xf(x)x \in Nx \subseteq P$ (i.e) $x \in P$ which is clearly a contradiction to $x \in J - P$. If $Ny \subseteq P$ then $Ny \subseteq J$ and this demand $sy = yf(y)y \in Ny \subseteq J$. Therefore, $y \in J$ (i.e) $n - nf(x)x \in J$. Now, since $x \in J$, $nf(x)x \in Nj \subseteq J$ [since $N = N_0$, every left ideal is an N -subgroup]. Therefore, $nf(x)x \in J$ and this implies $n \in J$ forcing $N = J$. The desired result now follows.

(ii) Let ' f ' be a mate function for N and let M be an N - subgroup of N . We observe that for all x in M , $f(x)xf(x) \in NMN \subseteq M$. [since M is an ideal]. This fact guarantees that we can define a map $g: M \rightarrow M$ such that $g(x) = f(x)xf(x)$. Clearly, g serves as a mate function for M .

We establish that $xM = xMx = Mx^2$ for all x in M .

Now for x, y in M , $xy \in xM \subseteq xN = xNx$ [by Theorem 5.7] $= xNxf(x)x \in xNMNx \subseteq xMx$ [since M is an ideal].

Therefore, $xM \subseteq xMx$. For the reverse inclusion, if $y \in M$, $xyx \in xMx \subseteq xNx = xN$ [by Theorem 5.7]

$$= xf(x)xN \in xNMN \subseteq xM. \text{ Hence } xMx \subseteq xM. \text{ Consequently, } xM = xMx \text{ for all } x \text{ in } M.$$

Again, $xyx \in xMx \subseteq xNx = Nx^2$ [by Theorem 5.7] $= Nx f(x)xx \in NMNx^2 \subseteq Mx^2$. Thus $xMx \subseteq Mx^2$. On the other hand, $yx^2 \in Mx^2 \subseteq Nx^2 = xNx$ [by Theorem 5.7] $= xf(x)xNx \in xNMNx \subseteq xMx$. Therefore, $Mx^2 \subseteq xMx$. Consequently, $xMx = Mx^2$ for all x in M .

Collecting all the sepieces, we get $xM = xMx = Mx^2$ for all x in M . Now, Theorem 5.7 guarantees that M , as a sub near-ring of N , is a γ near-ring.

Remark 5.9: It is worth noting that the existence of a mate function and the property $xN = xNx = Nx^2$ for all x in N are preserved under homomorphisms. Consequently, if N admits mate functions and is a γ near-ring, then any homomorphic image of N also does so.

Theorem 5.10: Let N be a γ near-ring with a mate function ' f ' and let $E \subseteq C(N)$. Then the following are equivalent.

- (i) N is subdirectly irreducible.
- (ii) None of the non-zero idempotents of N is a zero divisor.
- (iii) N is simple.

Proof: Since $E \subseteq C(N)$, we first observe that N is zero-symmetric.

(i) \Rightarrow (ii): Suppose N is sub directly irreducible. Let J be the set of all non-zero idempotents in N which are zero-divisors and suppose J is not empty. For any n in N , $(0:n)$ is a left ideal of N [by R(6)]. Since N is zero-symmetric, $(0:n)$ is an N -subgroup of N . [by R(7)]. Thus for every e in J , $(0:e)$ is an ideal of N [since N is a γ near-ring]. Let $I = \bigcap_{e \in J} (0:e)$. Since N is subdirectly irreducible, $I \neq \{0\}$ [by R(5)]. Let $x \in I - \{0\}$. Thus $xe = 0$ for all e in J . (1)

This implies $f(x)xe = f(x)0$ [by (1)] = 0 [since $N = N_0$] \Rightarrow $ef(x)x = 0$ [since $L = \{0\}$ and N has IFP by R(3)].

Therefore, $f(x)x \in J$. From (1), we get $x f(x)x = 0 \Rightarrow x = 0$ which is a contradiction. This contradiction guarantees that J is empty and (ii) follows

(ii) \Rightarrow (iii): Let M be a non-zero N -subgroup of N . Then M is an ideal of N and let $x (\neq 0) \in M$.

For any n in N , we have, $nx = nx f(x)x$. This implies $(n - nx f(x))x = 0$. Therefore, $(n - nx f(x))x f(x) = 0 f(x) = 0$. Hence by (ii), $n - nx f(x) = 0$. This implies $n = nx f(x) \in NMN \subseteq M$ [since M is an ideal of N]. Thus $N \subseteq M$. This shows that N has no nontrivial ideal of N . Hence N is simple [by R(9)].

(iii) \Rightarrow (i): Suppose N is simple. Obviously then N is subdirectly irreducible [by R (5)].

We conclude our discussion with the following structure theorem for γ near-rings.

Theorem 5.11 Let N be a γ near-ring with a mate function ' f ' and let $E \subseteq C(N)$. Then N is isomorphic to a subdirect product of simple near-rings.

Proof: By Theorem 1.62, p.26 of Pilz [3], N is isomorphic to a subdirect product of sub directly irreducible near-rings N_i 's say and each N_i is a homomorphic image of N under the projection map π_i . By Remark 5.9, N is isomorphic to a subdirect product of subdirectly irreducible γ near-rings N_i 's, each with a mate function. Obviously, each N_i is zero-symmetric and satisfies $E \subseteq C(N)$. Now, Theorem 5.10 demands that each N_i is simple and this completes the proof of the theorem.

REFERENCES

- [1] Akin Osman Atagiin, 'IFP ideals in Near-Rings', Hacettepe Journal of Mathematics and Statistics, Volume 39(1) 2010, 17 – 21.
- [2] J.R. Clay, The near-rings on groups of low order, Math.Z. 104 (1968), 364 – 371.
- [3] Gunter Pilz, Near-Rings, North Holland, Amsterdam, 1983.
- [4] J.L. Jat and S.C. Choudhary, 'On left bipotent Near-Rings' proceedings of the Edinburgh Mathematics Society 1979) 22, 99 – 107.
- [5] N.H. Mc.Coy, The Theory of Rings, Macmillan & Co, 1970.
- [6] G.Sugantha and R.Balakrishnan, 'Some Special Near-Rings' International Research Journal of Pure Algebra Vol-4(4), April 2014, 495 – 500.
- [7] S.Suryanarayanan and N. Ganesan, 'Stable and Pseudo Stable Near-Rings', Indian J. Pure and Appl. Math 19(12) December, 1988, 1206 – 1216.
- [8] S.Suryanarayanan and R.Balakrishnan, 'A near-ring N in which every N -subgroup is invariant' The Mathematics Education Vol XXXIII, No.3 Sept 1999.

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