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γ NEAR-RINGS

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ABSTRACT

In this paper we introduce the concept of γ near-rings. In [8], S. Suryanarayanan and R. Balakrishnan investigated a near-ring N in which every N-subgroup is invariant. Motivated by this concept, we probe into the properties of a near-ring N where every N-subgroup is an ideal. We discuss the properties of this newly introduced structure, obtain a complete characterization and a structure theorem for such near-rings.

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1. INTRODUCTION

Near-rings are generalized rings. If in a ring (N, +,·) with two binary operations'+' and '·', we ignore the commutativityof '+' and one of the distributive laws, (N, +, ·) becomes a near-ring. If we do not stipulate the left distributive law, (N, +, ·) becomes a right near-ring. Throughout this paper, N stands for a right near-ring (N, +, ·) with at least two elements.Obviously, 0n = 0 for all n in N, where '0' denotes the identity of the group (N, +). As in [3], a subgroup (M, +) of (N, +) is called (i) a left N-subgroup of N if MN⊆M, (ii) an N-subgroup of N if NM⊆M and (iii) an invariant N – subgroup of N if Msatisfies both (i) and (ii). Again in [3], a normal subgroup (I, +) of (N, +) is called (i) a left ideal if n(n′ + i) – nn′ ∈ I for all n, n′ ∈ N and i ∈ I (ii) a right ideal if IN ⊆ I and (iii) an ideal if I satisfies both (i) and (ii). In [4], N is said to be leftbipotent if Na = Na^2 for all a ∈ N. In [6], N is called a β_3 near – ring if xNy = yxN for all x, y ∈ N. An idea I of N is called (i) a prime ideal if for all ideals J, K of N, JK ⊆ I ⇒ J ⊆I or K ⊆ I. (ii) a completely semiprime ideal if for a ∈ N, a^2∈ I ⇒a ∈ I. (iii) an IFPideal [1], if for a, b ∈ N, ab∈ I ⇒an b∈ I for all n in N. (iv) a semiprime ideal if for all ideals J of N, J^2⊆ I ⇒ J ⊆ I. If {0} is a semiprime ideal, then N is called a semiprime near-ring [2.87, p.67 of Pilz [3]]. Also in [3], N is said to have property P_4 if for all ideals I of N, ab∈ I implies ba ∈ I for a, b in N. The concept of a mate function in N has been introduced in [7] with a view to handling the regularity structure with considerable ease. A map '*f*' from N into N is called a mate function for N if *x=xf(x)x* for all x in N. Also the existence of mate functions is preserved under homomorphisms. By identity 1of N, we mean only the multiplicative identity of N.

Basic concepts and terms used but left undefined in this paper can be found in Pilz [3].

2. NOTATIONS

(i) E denotes the set of all idempotents of N (e in N is called an idempotent if e^2 = e)
(ii) L denotes the set of all nilpotents of N (a in N is nilpotent if a^k = 0 for some positive integer k)
(iii) N_d = {n∈N/ n(x+y) = nx + ny for all x, y in N} – set of all distributive elements of N.
(iv) C(N) = {n∈N/ nx = xnfor all x in N} – centre of N.
(v) N_0 = {n∈ N / n0 = 0} – zero-symmetric part of N.

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3. PRELIMINARY RESULTS

We freely make use of the following results and designate them as R(1),R(2), ...etc

R(1) N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N (Problem 14, p.9 of [5])

R(2) If f is a mate function for N, then for every x in N, $xf(x)$, $f(x)x \in E$ and $Nx = Nf(x)x$, $xN = xf(x)N$ (Lemma 3.2 of [7])

R(3) If L={0} and N =N_0 then (i) xy = 0 ⇒yx = 0 for all x, y in N (ii) N has Insertion of Factors Property– IFP for short – i.e. for x, y in N, xy=0⇒xny=0 for all n in N. If N satisfies (i) and (ii) then N is said to have (∗, IFP) (Lemma 2.3 of [7])

R(4) N has strong IFP if and only if for all ideals I of N, and for x, y∈ N, xy∈ I ⇒ xny ∈ I for all n ∈ N (Proposition 9.2, p.289 of [3])

R(5) N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals is again nonzero (Theorem 1.60, p.25 of [3])

R(6) For any n in N, (0 : n) is a left ideal of N (1.43, p.21 of Pilz [3])

R(7) If N is zero-symmetric, then every left ideal is an N-subgroup (Proposition 1.34(b), p.19 of Pilz [3])

R(8) A zero-symmetric near-ring N has IFP if and only if (0: S) is an ideal where S is any non-empty subset of N (by 9.3, p.289 of [3])

R(9) A near-ring N is called simple if it has no non-trivial ideals of N. (By 1.36, p.19 of Pilz [3])

R(10) If N is a β_3 near-ring ,then every left N-subgroup of N is an N-subgroup of N (Proposition 5.3 (iv) of [6])

4. DEFINITION AND EXAMPLES OFγ NEAR-RINGS

In this section we define γ near-rings and give certain examples of this new concept.

Definition4.1: We say that a right near-ring N is a γ near-ring if every N-subgroup of N is an ideal of N.

Examples 4.2: (a) The near-ring (N, +, ·) defined on Klein's four group (N,+)with N={0,a,b,c} where '·'is defined as per scheme 22, p.408 of Pilz [3]

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

is a γ near – ring.

(b) Let (N, +) be the Klein's four group as in (a) above. If multiplication is defined as per scheme 11, p.408 of Pilz [3],

·	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

then N is not a γ near-ring, as the N-subgroup {0, a} is not an ideal.

5. PROPERTIES OF γ NEAR-RINGS

In this section we prove certain important properties of γ near-rings and give a complete characterization of such near-rings.

Proposition 5.1: Let N be a γ near-ring. If N is a β_3 near-ring with identity and N = N_d, then every left N-subgroup is an ideal.

Proof: Since N is a γnear-ring, every N-subgroup of N is an ideal. (1)

Let M be any left N-subgroup of N. Since N is a β_3 near-ring, by R(10), M is an N-subgroup of N. This implies M is an ideal [by (1)].

Therefore, every left N-subgroup of N is an ideal.

Proposition 5.2: Let N be a γnear-ring. Then every left N-subgroup of N is invariant.

Proof: Let M be an N-subgroup of N. Since N is a γ near-ring, M becomes an ideal of N. Now, the desired result follows from the definition of right ideal.

Remark 5.3: The converse of Proposition 5.2 is not valid. For example, consider the near-ring (N, +,·) where (N,+) is the usual group of integers modulo 6 and where '·' is defined as per scheme 24,p.408 of Pilz [3]

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

We observe that, every N-subgroup of N is invariant. However N is not a γ near-ring, since the N-subgroup {0, 3} is not an ideal.

Proposition 5.4: Let N be aγnear-ring which admits a mate function '*f*'. Then
(i) for all N – subgroups A and B of N, A ∩ B = AB.
(ii) Nx ∩ Ny = Nxy for all x, y in N.

Proof: (i) Let A and B be two N-subgroups of N. Since N is a γ near-ring, A and B are ideals of N. Hence AN⊆ A and BN ⊆ B.

Now, for x ∈ A and y ∈ B, xy ∈ AN ⊆ A.Therefore, AB ⊆ A.

Also, xy∈ NB ⊆ B. Hence AB ⊆ B. Consequently,

$$AB \subseteq A \cap B \quad (1)$$

On the other hand, if z ∈ A ∩ B, then since '*f*' is a mate function for N, z = z*f*(z)z ∈ (AN)B ⊆ AB. Thus

$$A \cap B \subseteq AB \quad (2)$$

Combining (1) and (2) A ∩ B = AB for all N-subgroups A, B of N.

(ii) Let x, y∈ N. Then by taking A = Nx and B = Ny in (i) we get,

$$Nx \cap Ny = NxNy \quad (3)$$

Again by taking A = Nx and B = N in (i) we get, Nx = Nx∩N = NxN. Therefore,

$$Nxy = NxNy \quad (4)$$

From (3) and (4), we get Nx∩Ny = Nxy for all x, y in N.

We furnish below a characterization theorem forγnear-rings.

Theorem 5.5: Let N be a near-ring which admits a mate function '*f*'. Then the following are equivalent.
(i) N is a γ near-ring.
(ii) Every N-subgroup is a completely semiprime ideal of N.
(iii) Every N-subgroup is an IFP ideal.

Proof:

(i) ⇒ (ii): Let M be any N-subgroup of N. Since N is a γ near-ring, M becomes an ideal of N. Let $x^2 \in M$. Now, since 'f' is a mate function for N, $x = xf(x)x \in Nx = Nx \cap Nx = Nx^2$ [by Proposition 5.4 (ii)] $\subseteq NM \subseteq M$. Therefore, $x \in M$ and (ii) follows.

(ii)⟹ (iii): Let M be any N-subgroup of N and let $xy \in M$. Now, $(yx)(yx) = y(xy)x \in NMN \subseteq M$ by (ii). Thus we have $(yx)^2 \in M$ and (ii) implies $(yx) \in M$.For all n in N, $(xny)^2 = (xny)(xny) = xn(yx)ny \in NMN \subseteq M$ and again (ii) guarantees that $xny \in M$ and (iii) follows.

(iii)⟹ (i) Obvious.

Proposition 5.6: Let N be a γnear-ring and let N admit a mate function 'f'. Then we have

(i) N is left bipotent.
(ii) N has property P_4.
(iii) N has strong IFP.
(iv) N is a semiprime near-ring.

Proof: Let N be a γnear-ring and let M be an N-subgroup of N. Then M is an ideal of N.

(i) Since N admits a mate function 'f', we have by Proposition 5.4 (ii), $Nx = Nx \cap Nx = Nx^2$.It follows that $Nx = Nx^2$ and hence (i) follows.

(ii) Let I be an ideal of N. Let $xy \in I$. Now, $(yx)^2 = (yx)(yx) = y(xy)x \in NIN \subseteq IN \subseteq I$.Therefore, $(yx)^2 \in I$.This implies $yx \in I$ [by Proposition 5.5 (ii)]. Consequently, N has property P_4.

(iii) Let $xy \in I$.Then $yx \in I$ [by (ii)]. Now, $yxn \in IN \subseteq I$ for all n in N. This implies $y(xn) \in I$. Therefore, $xny \in I$ [by (ii)] and (iii) follows.

(iv) Let I be any ideal of N such that $I^2 \subseteq M$. Now, for $x \in I$, since 'f' is a mate function for N, $x = xf(x)x \in INI \subseteq I^2 \subseteq M$. Hence $I \subseteq M$.Thus I is a semiprimeideal. In particular, {0} is a semiprimeideal of N. Therefore, N is a semiprime near – ring.

We furnish below another characterization ofγ near-rings.

Theorem 5.7: Let N admit a mate function 'f' and let $E \subseteq C(N)$. Then N is aγ near-ring if and only if $xN = xNx = Nx^2$ for all x in N.

Proof: Since $E \subseteq C(N)$we first observe that N is zero-symmetric .

For the 'only if' part, we see that for every x in N, as N is a γ near-ring, Nx, being an N-subgroup, is an ideal of N.

Therefore,
$(Nx)N \subseteq Nx$ (1)
and
$N(Nx) \subseteq Nx$ (2)

Hence for any n in N, since 'f 'is a mate function for N, $xn = (xf(x)x)n = x(f(x)xn) = xn'x$ for some n'in N [by(1)].

Therefore $xn \in xNx$.Thus $xN \subseteq xNx$.Obviously $xNx \subseteq Nx$ holds. Consequently we have, $xN = xNx$ for all x in N. Again, for any n in N, $nx^2 = n(xf(x)x)x = x(f(x)nx)x$ [since $E \subseteq C(N)$] $= x(n''x)x$ for some n'' in N [by (2)] $\in xNx$. Thus

$Nx^2 \subseteq xNx$ (3)

For the reverse inclusion, we have for any n in N, $xnx = xf(x)xnx = xnf(x)xx$ [since $E \subseteq C(N)$] $\in xNxx$[by (2)] = $xf(x)Nxx$[by R(2)] $= Nxf(x)xx$ [since $E \subseteq C(N)$] $= Nxx$[since 'f' is a mate function for N] $= Nx^2$. Therefore, $xnx \in Nx^2$. Consequently,
$xNx \subseteq Nx^2$ (4)

Combining (3) and (4), $xNx = Nx^2$ for all x in N (5)

Collecting all these pieces we get, $xN = xNx = Nx^2$ for all x in N

For the 'if' part, first let us show that N has (∗, IFP). For any x in N, since '*f*' is a mate function for N, $x = xf(x)x \in xNx = Nx^2$ [by assumption (5)]. Therefore $x = n_1x^2$ for some n_1 in N. This yields that $x^2 = 0 \Rightarrow x = 0$ and R(1) guarantees L={0}. Also, since $N=N_0$, from R(3), we see that N has (∗,IFP). (6)

We have, by R(2), for any x∈N,

$Nf(x)x = Nx$ (7)

Let S = {n – ne/n∈ N}. We claim that (0: S) = Nx. Since (n – ne)e = 0 for all n in N, we get(n – ne)Ne = {0}[since N has (∗, IFP)].

Taking e = *f*(x)x,we get,(n – ne)Nx = {0} [by (7)]. Consequently,

$Nx \subseteq (0: S)$ (8)

To prove the reverse inclusion, we consider an arbitrary y in (0: S).Therefore yS = {0}. Since *f*(x)x ∈ E, we have $y(y - yf(x)x) = 0 \Rightarrow yf(x)x(y - yf(x)x) = 0$[since N has (∗,IFP)] and {y(y-yf(x)x)} –{yf(x)x(y-yf(x)x)}= 0.

Hence $(y - yf(x)x)^2 = 0$. Since L= {0}, R(1) guarantees that y – y*f*(x)x = 0. Thus y = y*f*(x)x ∈N *f*(x)x =Nx[by R(2)].

Therefore y∈Nx. This implies

$(0: S) \subseteq Nx$ (9)

From (8) and (9), we get, Nx =(0: S)for all x in N.

Now, R(8) guarantees that Nx is an ideal. If M is any N-subgroup of N, then we have $M = \sum_{x\in M} Nx$. It follows that M is an ideal and hence N becomes a γ near-ring.

Theorem 5.8 Let N be a γ near-ring which admits a mate function '*f*'' and let E⊆ C(N).Then

(i) any prime ideal of N is a maximal ideal.
(ii) every N-subgroup of N is a γ near-ring in its own right.

Proof: Let N be a γ near-ring. Since E⊆C(N), N is zero-symmetric. Further N has (∗, IFP). [by (6) of Theorem5.7]
(i) Let P be a prime ideal of N. Let J be an ideal of N such that J≠P and that P ⊂ J ⊂ N.Let x ∈ J – P.For x in N, since '*f* ' is a mate function for N, x = x*f*(x)x =*f*(x)xx [since E⊆C(N)]. Thus for all n in N, $nx = nf(x)x^2$ and this implies (n – n*f*(x)x)x = 0. Since N has (∗, IFP), we get (n – n*f*(x)x)zx = 0. And z(n – n*f*(x)x)zx = z.0 = 0 [since $N=N_0$] for all z ∈ N. Consequently, N(n – n*f*(x)x)Nx ={0}. If we let y = n – n*f*(x)x, then NyNx ={0}⊆ P. Also, since N is a γ near-ring, Nx, Ny are ideals in N. Since P is prime, we get Ny⊆ P or Nx⊆ P. If Nx⊆P then x = x*f*(x)x∈Nx ⊆ P(i.e) x ∈ P which is clearly a contradiction to x∈ J – P. If Ny⊆P then Ny⊆ J and this demand sy = y*f*(y)y∈Ny ⊆ J. Therefore, y ∈ J (i.e)n – n*f*(x)x ∈J. Now, since x ∈J, n*f*(x)x ∈NJ⊆J [since N= N_0, every left ideal is an N-subgroup]. Therefore, n*f*(x)x ∈J and this implies n ∈J forcing N = J. The desired result now follows.

(ii) Let '*f*' be a mate function for N and let M be an N – subgroup of N. We observe that for all x in M, *f*(x)x*f*(x) ∈NMN⊆M. [since M is an ideal]. This fact guarantees that we can define a map g: M→M such that g(x)=*f*(x)x*f*(x). Clearly, g serves as a mate function for M.

We establish that $xM=xMx= Mx^2$ for all x in M.

Now for x, y in M, xy∈xM⊆xN=xNx [by Theorem 5.7] =xNx*f*(x)x∈xNMNx⊆xMx[since M is an ideal].

Therefore, xM ⊆xMx.For the reverse inclusion, if y ∈M, xyx ∈xMx⊆xNx=xN [by Theorem 5.7]

=x*f*(x)xN∈xNMN⊆xM. Hence xMx⊆xM. Consequently, xM=xMxforall x in M.

Again, xyx ∈ xMx⊆ xNx= Nx^2 [by Theorem 5.7] =N*x**f*(x)xx∈ $NMNx^2 \subseteq Mx^2$. Thus $xMx \subseteq Mx^2$.On the other hand, $yx^2 \in Mx^2 \subseteq Nx^2$=xNx[by Theorem 5.7] =x*f*(x)xNx∈xNMNx⊆xMx. Therefore, $Mx^2 \subseteq xMx$. Consequently, $xMx = Mx^2$for all x in M.

Collecting all the sepieces, we get $xM=xMx=Mx^2$ for all x in M. Now, Theorem 5.7 guarantees that M, as a sub near-ring of N, is a γ near-ring.

Remark 5.9: It is worth noting that the existence of a mate function and the property xN=xNx= Nx^2 for all x in N are preserved under homomorphisms. Consequently, if N admits mate functions and is a γ near-ring, then any homomorphic image of N also does so.

Theorem 5.10: Let N be a γnear-ring with a mate function '*f*' and let E⊆ C(N).Then the following are equivalent.
(i) N is subdirectly irreducible.
(ii) None of the non-zero idempotents of N is a zero divisor.
(iii)N is simple.

Proof: Since E⊆C(N)**,** we first observe that N is zero-symmetric.
(i) ⇒ (ii): Suppose N is sub directlyirreducible. Let J be the set of all non-zero idempotents in N which are zero-divisors and suppose J is not empty. For any n in N, (0:n) is a left ideal of N [by R(6)].Since N is zero-symmetric, (0:n) is an N-subgroup of N.[by R(7)]. Thus for every e in J, (0: e) is an ideal of N [since N is a γ near-ring]. Let $I = \bigcap_{e \in J}(0 : e)$. Since N is subdirectly irreducible, I≠{0} [by R(5)]. Let $x \in I - \{0\}$**.** Thus xe = 0 for all e in J. (1)

This implies *f*(x)xe=*f*(x)0[by(1)] = 0 [since N= N_0]⇒ e*f*(x)x=0 [since L={0} and N has IFP by R(3)].

Therefore, *f*(x)x ∈J. From (1), we getx*f*(x)x=0⇒x=0which is a contradiction. This contradiction guarantees that J is empty and (ii) follows

(ii)⇒(iii): Let M be a non-zero N-subgroup of N. Then M is an ideal of N and let x($\neq 0$) ∈M.

For any n in N, we have, nx=nx*f*(x)x. This implies (n – nx*f*(x))x=0. Therefore, (n – nx*f*(x))x*f*(x)=0*f*(x)=0. Hence by (ii), n – nx*f*(x) =0. This implies n=nx*f*(x)∈NMN⊆M [since M is an ideal of N]. Thus N⊆M.This shows that N has no nontrivial ideal of N. Hence N is simple [by R(9)].

(iii) ⇒ (i): Suppose N is simple. Obviously then N is subdirectly irreducible [by R (5)].

We conclude our discussion with the following structure theorem for γ near-rings.

Theorem 5.11 Let N be a γ near-ring with a mate function '*f*' and let E ⊆ C(N). Then N is isomorphic to a subdirect product of simple near-rings.

Proof: By Theorem 1.62, p.26 of Pilz [3], N is isomorphic to a subdirect product of sub directly irreducible near-rings N_i's say and each N_i is a homomorphic image of N under the projection map π_i. By Remark 5.9, N is isomorphic to a subdirect product of subdirectly irreducible γ near-rings N_i's, each with a mate function. Obviously, each N_i is zero-symmetric and satisfies E ⊆ C(N).Now, Theorem 5.10 demands that each N_i is simple and this completes the proof of the theorem.

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