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FIXED POINT THEOREM IN MENGER SPACE

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ABSTRACT

In this paper, we have accomplished the task of generalizing the fixed point theorems due to Som [3], Taskovic [4] Mukherjee [1] in the context of semi-compatibility and weak compatibility in Menger space and also it has been applied to prove common fixed point theorems for three, four and sequence of mappings. A coincidence point theorem is also established for a multi-valued mapping satisfying a generalized condition of Hausdorff distance function in (T, f, x)-orbitally complete Menger space. This theorem is generalized result of Tiwari and Shrivastava [5] and Singh [2].

BASIC PRELIMINARIES

Following notations and definitions will be used in this paper.

CL(X) = {A: A is non-empty closed subset of X}.

Definition 1: Let T be a multi-valued mapping on a Menger space (X, F, Δ) and $x_0 \in X$. A sequence $\{x_n\}$ in X said to be an **orbit of T at x_0** denoted by $o(T, x_0)$ if $x_{n-1} \in T^n(x_0)$, i.e., $x_n \in Tx_{n-1}$, $\forall\, n \in N$.

If T is a self mapping then the sequence $\{x_n\}$, $x_n = T^{\,n-1}(x_0)$, $\forall\, n \in N$, is the **orbit of T at x_0**.

Definition 2: A Manger space (X, F,Δ) is said to be **T-orbitally complete** iff every cauchy sequence of the form $\left(x_{n_i} : x_{n_i} \in Tx_{n_{i-1}}\right)$ converges in X.

Definition 3: Let S and T be a single-valued mappings and multi-valued mapping respectively on a Menger space (X, F, Δ). A Menger space (X, F, Δ) is said to be **(T, S, X)-orbitally complete** iff every cauchy sequence of the form $\left(Sx_{n_i} : Sx_{n_i} \in Tx_{n_{i-1}}\right)$ converges in X.

Definition 4: A multi-valued mapping T in Menger space (X, F, Δ) is said to be **asymptotically regular at x_0**, if for each sequence $\{x_n\}$ in X, $x_n \in Tx_{n-1}$ and $F_{x_n, x_{n+1}}(t) \to 1$ as $n \to \infty$, for all $t > 0$

Definition 5: Let S and T be a single-valued mappings and multi-valued mapping respectively on a Menger space (X, F, Δ). A point x in X is said to be a **coincidence point** of S and T if $Sx \in Tx$.

Definition 6: Self mappings A and S of a menger space (X, F,Δ) are called **semi-compatible** if $F_{AS_n, Su}(\varepsilon) \to 1$, for all $\varepsilon > 0$,whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$, for some u in X.

Proposition 7: Let A and S be self mappings on a Menger space (X, F,Δ) with $\Delta(a, a) \geq a$, for all $a \in [0, 1]$. If S is continuous then (A, S) is semi-compatible iff (A, S) is compatible.

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Proof: Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$, As S is continuous, we have $SAx_n \to Su$.

Suppose that (A, S) is semi-compatible, then for given (ε, λ), we have a positive integer $N_0(\varepsilon, \lambda)$ such that

$$F_{AS_n, Su}\left(\frac{\varepsilon}{2}\right) \geq 1-\lambda \quad \text{and} \quad F_{SA_n, Su}\left(\frac{\varepsilon}{2}\right) \geq 1-\lambda, \ \forall\, n > N_0.$$

Now

$$\begin{aligned} F_{ASx_n, SAx_n}(\varepsilon) &\geq \Delta\left(F_{ASx_n, Su}\left(\frac{\varepsilon}{2}\right), F_{SAx_n, Su}\left(\frac{\varepsilon}{2}\right)\right) \\ &\geq \Delta(1-\lambda, 1-\lambda), \qquad \forall\, n \geq N_0 \\ &\geq 1-\lambda, \qquad \forall\, n \geq N_0. \end{aligned}$$

Hence the pair (A, S) is compatible.

Conversely, let the pair (A, S) be compatible. Then for given (ε, λ), we have a positive integer $N_0(\varepsilon, \lambda)$ such that

$$F_{ASx_n, SAx_n}\left(\frac{\varepsilon}{2}\right) \geq 1-\lambda, \ F_{SAx_n, Su}\left(\frac{\varepsilon}{2}\right) \geq 1-\lambda, \ \forall\, n \geq N_0.$$

Now,

$$\begin{aligned} F_{ASx_n, Su}(\varepsilon) &\geq \Delta\left(F_{ASx_n, SAx_n}\left(\frac{\varepsilon}{2}\right), F_{SAx_n, Su}\left(\frac{\varepsilon}{2}\right)\right) \\ &\geq \Delta(1-\lambda, 1-\lambda) \\ &\geq 1-\lambda; \ \forall\, n \geq N_0. \end{aligned}$$

Hence $ASx_n \to Su$, i.e. (A, S) is semi-compatible.

Definition 8: Two self mappings A and S of a menger space (X, F, Δ) are said to be **reciprocally continuous** if

$$\lim_{n\to\infty} ASx_n = At \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t; \qquad \text{for some } t \in X$$

If A and S are both continuous, then they are obviously reciprocally continuous but the converse is not true.

MAIN RESULTS

The following theorem is given by R. Tiwari and S.K. Shrivastava [5]

Theorem 9: Let T be a multi-valued mapping from a metric space X to CL(X). If there exists f: X → X such that $TX \subseteq fX$, for each x, y $\in$ X, and

$$H(Tx, Ty) \leq \phi\begin{pmatrix} \alpha d(fx, fy) + \beta\left[D(fx, Tx) + D(fx, Ty)\right] + \gamma\left[D(fx, fy) + D(fy, Tx)\right] \\ \sigma \max\left\{d(fx, fy), \frac{1}{2}\left[D(fx, Tx) + D(fy, Ty)\right], \frac{1}{2}\left[D(fx, Ty) + D(fy, Tx)\right]\right\} \end{pmatrix}$$

where $\max\{\alpha + 2\gamma + \sigma, \beta + \gamma + \sigma\} \leq 1$, $\alpha, \beta, \gamma \geq 0$, $0 < \sigma \leq 1$, $\phi(t) < qt$ for each $t > 0$ for some fixed $0 < q < 1$, $\phi \in \psi$ and there exists an $x_0 \in X$ such that T is asymptotically regular at x_0, and X is (T, f, x)-orbitally complete, then T and f have a coincidence point.

Taking the clue from above theorem 9, we prove the following theorem

Theorem 10: Let (X, F, Δ) be a Menger space, where $\Delta(a, b) = \min\{a, b\}$, for all a, b,$\in$ [0, 1] and T be a multi-valued mapping from X to CL (X). If there exists mapping S: X → X such that

(a) $TX \subseteq SX$, for each x, y $\in$ X, and

(b) $$F_{Tx,Ty}(\phi t) \geq \min\left(\begin{array}{l} F_{Sx,Sy}(t), F_{Sx,Tx}(t), F_{Sy,Ty}(t), F_{Sx,Ty}(t), F_{Sy,Tx}(t), \\ \max\left\{ F_{Sx,Sy}(t), F_{Sx,Tx}(\alpha t), F_{Sy,Ty}((2-\alpha)t), F_{Sy,Ty}(\beta t), F_{Sy,Tx}((2-\beta)t) \right\} \end{array} \right)$$

for all $t > 0$, $\alpha, \beta \in (0, 1)$,

(c) $\phi(t) < qt$, $\forall\, t > 0$, $0 < q < 1$, $\phi \in \Phi$,

(d) There exists an $x_0 \in X$ such that T is asymptotically regular at x_0,

(e) (T, S, x) - orbitally complete.

Then T and S have a coincidence point.

Proof: Choose $x_0 \in X$ satisfying (a). We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows:

Since $TX \subset SX$, Choose $y_1 = Sx_1 \in Tx_0$. If $Tx_0 = Tx_1$, Choose $y_2 = Sx_2 \in Tx_1$ such that $y_1 = y_2$, If $Tx_0 \neq Tx_1$, from the definition of Hausdorff distance one can choose $y_2 = Sx_2 \in Tx_1$ such that

$$F_{y_1, y_2}(t) \geq F_{Tx_0, Tx_1}(t)$$

In general, Choose $y_{n+2} = Sx_{n+2} \in Tx_{n+1}$, such that $y_{n+1} = y_{n+2}$ if $Tx_n = Tx_{n+1}$ and $F_{y_{n+1}, y_{n+2}}(t) \geq F_{Tx_n, Tx_{n+1}}(t)$ Otherwise.

We wise to show that $\{y_n\}$ is cauchy. For this it is sufficient to show that $\{y_{2n}\}$ is cauchy.

Suppose on the contrary that $\{y_{2n}\}$ is not cauchy. Then there is an $\varepsilon > 0$ such that for each integer 2k, k = 0, 1, 2, ... there exists even integers 2nk and 2mk with $2k < 2nk < 2mk$ such that

$$F_{y_{2nk}, y_{2mk}}(\varepsilon) < 1-\lambda. \tag{1}$$

Let for each integer 2k, 2mk be the least positive integer exceeding 2nk satisfying (1). Then,

$$F_{y_{2nk}, y_{2mk}}(\varepsilon) \geq 1-\lambda \tag{2}$$

$$F_{y_{2nk}, y_{2mk}}(\varepsilon) < 1-\lambda.$$

As such, for each even integer 2k, we have

$$1-\lambda > F_{y_{2nk}, y_{2mk}}(\varepsilon) \geq F_{y_{2nk}, y_{2mk}}(\varepsilon) \geq \Delta\left(F_{y_{2nk}, y_{2mk-2}}\left(\frac{\varepsilon}{3}\right), F_{y_{2mk-2}, y_{2mk-1}}\left(\frac{\varepsilon}{3}\right), F_{y_{2mk-1}, y_{2mk}}\left(\frac{\varepsilon}{3}\right) \right)$$

So by (2) and $k \to \infty$, we get

$$\lim_{n\to\infty} F_{y_{2nk}, y_{2mk}}(\varepsilon) = 1-\lambda. \tag{3}$$

Now, using (3) in the triangle inequality

$$F_{y_{2nk}, y_{2mk-1}}(\varepsilon) \geq \Delta\left(F_{y_{2nk}, y_{2mk}}\left(\frac{\varepsilon}{2}\right), F_{y_{2mk}, y_{2mk-1}}\left(\frac{\varepsilon}{2}\right) \right)$$

and

$$F_{y_{2nk+1}, y_{2mk-1}}(\varepsilon) \geq \Delta\left(F_{y_{2nk+1}, y_{2nk}}\left(\frac{\varepsilon}{3}\right), F_{y_{2nk}, y_{2mk}}\left(\frac{\varepsilon}{3}\right), F_{y_{2mk}, y_{2mk-1}}\left(\frac{\varepsilon}{3}\right) \right)$$

Taking $k \to \infty$

$$F_{y_{2nk+1}, y_{2mk-1}}(\varepsilon) \geq \Delta\,[1-\lambda, 1] = 1-\lambda \text{ and} \tag{4}$$

$$F_{y_{2nk+1}, y_{2mk-1}}(\varepsilon) \geq \Delta\,[1-\lambda, 1, 1] = 1-\lambda. \tag{5}$$

Then,

$$F_{y_{2nk},y_{2mk}}(\phi t) \geq \Delta\left(F_{y_{2nk},y_{2nk+1}}\left(\frac{\phi t}{2}\right), F_{y_{2nk+1},y_{2mk}}\left(\frac{\phi t}{2}\right)\right)$$

$$\geq \Delta\left(F_{y_{2nk},y_{2nk+1}}\left(\frac{\phi t}{2}\right), F_{Tx_{2nk+1},Tx_{2mk}}\left(\frac{\phi t}{2}\right)\right)$$

$$\geq \Delta\left(F_{y_{2nk},y_{2nk+1}}\left(\frac{\phi t}{2}\right), \min\left(\begin{array}{l} F_{Sx_{2nk+1},Sx_{2mk}}(2t), F_{Sx_{2nk+1},Tx_{2nk+1}}(2t), F_{Sx_{2mk},Tx_{2mk}}(2t), \\ F_{Sx_{2nk+1},Tx_{2mk}}(2t), F_{Sx_{2mk},Tx_{2nk+1}}(2t), \\ \max\left(\begin{array}{l} F_{Sx_{2nk+1},Sx_{2mk}}(2t), F_{Sx_{2nk+1},Tx_{2nk+1}}(2(1-r)t), F_{Sx_{2nk+1},Tx_{2nk+1}}(2(1+r)t), \\ F_{Sx_{2nk+1},Tx_{2mk}}(2(1-q)t), F_{Sx_{2mk},Tx_{2nk+1}}(2(1+q)t) \end{array}\right) \end{array}\right)\right)$$

Putting $\beta = 1-q$, $\alpha = 1-r$, $q, r \in (0, 1)$

$$\geq \Delta\left(F_{y_{2nk},y_{2nk+1}}\left(\frac{\phi t}{2}\right), \min\left(\begin{array}{l} F_{y_{2nk+1},y_{2mk}}(2t), F_{y_{2nk+1},y_{2nk+2}}(2t), F_{y_{2mk},y_{2mk+1}}(2t), \\ F_{y_{2nk+1},y_{2mk+1}}(2t), F_{y_{2mk},y_{2nk+2}}(2t), \\ \max\left(\begin{array}{l} F_{y_{2nk+1},y_{2mk}}(2t), F_{y_{2nk+1},y_{2nk+2}}(2(1-r)t), F_{y_{2nk+1},y_{2nk+1}}(2(1+r)t), \\ F_{y_{2nk+1},y_{2mk+1}}(2(1-q)t), F_{y_{2mk},y_{2nk+1}}(2(1+q)t) \end{array}\right) \end{array}\right)\right)$$

Since ϕ is upper semi-continuous, taking the limit as $k \to \infty$

$1-\lambda \geq \Delta\,(1, (1 - \lambda), 1, 1, (1 - \lambda), (1 - \lambda), \max\{(1 - \lambda), 1, 1, (1 - \lambda), (1 - \lambda)\})$

$1-\lambda \geq 1-\lambda,$

which is a contraction.

Thus $\{y_n\}$ is a cauchy sequence. Since SX is (T, S, x_0)-orbitally complete, $\{y_n\}$ converges to a point u in X. Hence there exists a point z in SX such that u= Sz. Then,

$$F_{\mathrm{Sz,Tz}}(\phi t) \geq \Delta\left(F_{Sz,Sx_{n+1}}\left(\frac{\phi t}{2}\right), F_{Sx_{2n+1},Tz}\left(\frac{\phi t}{2}\right)\right)$$

$$\geq \Delta\left(F_{Sz,Sx_{n+1}}\left(\frac{\phi t}{2}\right), F_{Tx_n,Tz}\left(\frac{\phi t}{2}\right)\right)$$

$$\geq \Delta\left(F_{Sz,Sx_{n+1}}\left(\frac{\phi t}{2}\right), \min\left(\begin{array}{l} F_{Sx_n,Sz}(2t), F_{Sx_n,Tx_n}(2t), F_{Sz,Tz}(2t), \\ F_{Sx_n,Tz}(2t), F_{sz,Tx_n}(2t), \\ \max\left(\begin{array}{l} F_{Sx_n,Sz}(2t), F_{Sx_n,Tx_n}(2(1-r)t), F_{Sz,Tz}(2(1+r)t), \\ F_{Sx_n,Tz}(2(1-q)t), F_{Sz,Tx_n}(2(1+q)t) \end{array}\right) \end{array}\right)\right)$$

Taking limit $n \to \infty$, we have

$$F_{Sz,Tz}(\phi t) \geq \Delta\left(1, \min\left(\begin{array}{l} 1, F_{Sz,Tz}(2t), F_{Sz,Tz}(2t), \\ F_{Sz,Tz}(2t), F_{sz,Tz}(2t), \\ \max\left(\begin{array}{l} 1, F_{Sz,Tz}(2(1-r)t), F_{Sz,Tz}(2(1+r)t), \\ F_{Sz,Tz}(2(1-q)t), F_{Sz,Tz}(2(1+q)t) \end{array}\right) \end{array}\right)\right)$$

$$\geq \Delta\left(1, \min\left(1, F_{Sz,Tz}(2t), 1\right)\right)$$

$$F_{Sz,Tz}(\phi t) \geq F_{Sz,Tz}(2t)$$

$$F_{Sz,Tz}(t) \geq F_{Sz,Tz}\left(\phi^{-1} t\right)$$

Hence $Sz \in Tz$.

z is a coincidence point of S and T.

APPLICATIONS

In this section we study the existence of fixed point for multi-valued and self-mappings in a metric space (X, d) using the results in main result.

Theorem 11: Let (X, d) be a complete metric space and $T:(X, d) \to (CL(X), d_H)$. If there exists a mapping $S: (X, d) \to (X, d)$ such that

(a) $TX \subseteq SX$, for each $x, y \in X$, and

(b) $d_H(Tx, Ty) \leq \phi \max (d(Sx, Sy), d_H(Sx, Tx), d_H(Sy, Ty), d_H(Sx, Ty), d_H(Sy, Tx)), \min \{d(Sx, Sy),$ $\frac{1}{2}[d_H(Sy, Ty) + d_H(Sx, Tx)], \frac{1}{2}[d_H(Sx, Ty) + d_H(Sy, Tx)]\},$

where $\phi(t) < qt$ for each $t > 0$, $0 < q > 1$, $\phi \in \Phi$ and

(c) there exists an $x_0 \in X$ such that T is asymptotically regular at x_0,

(d) X is (T, S, X) - orbitally complete.

Then T and S have a coincidence point.

Proof: If we define $F: X \times X \to D^+$ by $F_{A, B}(t) = H(t - d_H(A, B))$, where $A, B \in CL(X)$, then the space (X, F, min) with a t-norm $\Delta = \min$ is a Menger space and the topology induced by the metric d coincides with the topology τ. And for any $Tx, Ty \in CL(X)$, we have

$$\begin{aligned} F_{Tx, Ty}(\phi t) &\geq H[\phi t - d_H(Tx, Ty)] \\ &\geq H[\phi t - \max \{d(Sx, Sy), d_H(Sx, Tx), d_H(Sx, Ty), \\ &\quad d_H(Sx, Ty), d_H(Sx, Tx)\}, \min \{d(Sx, Sy), \\ &\quad \tfrac{1}{2}[d_H(Sx, Tx) + d_H(Sx, Ty)], \tfrac{1}{2}[d_H(Sx, Ty) + d_H(Sy, Tx)]\}] \\ &= H[t - \max \{d_1, d_2, d_3, d_4, d_5, \min \{d_6, \tfrac{1}{2}(d_7 + d_8), \tfrac{1}{2}(d_9 + d_{10})\}] \end{aligned}$$

where, $d_1 = d(Sx, Sy)$, $d_2 = d_H(Sx, Tx)$, $d_3 = d_H(Sx, Ty)$, $d_4 = d_H(Sx, Ty)$,

$d_5 = d_H(Sx, Tx)$, $d_6 = d(Sx, Sy)$, $d_7 = d_H(Sx, Tx)$, $d_8 = d_H(Sx, Ty)$,

$d_9 = d_H$ (Sx, Ty), $d_{10} = d_H$ (Sy, Tx)

$$= H\,[\min\{(t - d_1), (t - d_2), (t - d_3), (t - d_4), (t - d_5), \max\{(t - d_6), (t - \tfrac{1}{2}(d_7 - d_8)), (t - \tfrac{1}{2}(d_9 + d_{10}))\}$$

$$= \min\{H(t - d_1), H(t - d_2), H(t - d_3), H(t - d_4), H(t - d_5), \max\{H(t - d_6), H(\alpha t - d_7), H((2 - \alpha)t - d_8), H(\beta t - d_9), H((2 - \beta)t - d_{10})\}]\text{ for some } \alpha, \beta \in (0, 2)$$

$$= \min\{F_{Sx,Sy}(t), F_{Sx,Tx}(t), F_{Sy,Ty}(t), F_{Sx,Ty}(t), F_{Sy,Tx}(t), \max\{F_{Sx,Sy}(t), F_{Sx,Tx}(\alpha t), F_{Sy,Ty}((2 - \alpha)t), F_{Sx,Ty}(\beta t), F_{Sy,Tx}((2 - \beta)t)\}$$

Thus Theorem 10 follows from Theorem 11 immediately.

Hence there exists a coincident point.

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