

**n<sup>th</sup> FINITE INTEGRALS INVOLVING PRODUCT  
OF MULTIVARIABLE JACOBI POLYNOMIAL AND H-FUNCTION OF SEVERAL VARIABLES**

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**ABSTRACT**

**I**n the present paper few finite integrals involving products of multivariable Jacobi polynomial and H function of several variables of generalized arguments have been evaluated. These integrals have been utilized to establish the expansion formula for H[Z<sub>1</sub>, Z<sub>2</sub>, ..., Z<sub>r</sub>] function in series involving product of multivariable Jacobi polynomials since H function of multivariable quite general function in nature. On specializing the parameters of the function involved in results many new as well as known relations may be obtained as particular case.

**Key words:** Multivariable Jacobi Polynomials.

**1. INTRODUCTION**

The classical Jacobi polynomial for one variable may be defined as [1; equation (1), p. 254]

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n; 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right) \quad (1.11)$$

Two variable, three variable and multivariable Jacobi polynomial defined by Shrivastava [2, equ. (1.3) (1.5) (1.7) pg 159-161] in the following manner

$$P_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2)}(x_1, x_2) = \frac{(1+\alpha_1)_n (1+\alpha_2)_n}{(n!)^2} {}_3F_2\left(-n; 1+\alpha_2 + \beta_2 + n; 1+\alpha_1 + \beta_1 + n; 1+\alpha_2; 1+\alpha_1; \frac{1-x_2}{2}, \frac{1-x_1}{2}\right) \quad (1.12)$$

$$P_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)}(x_1, x_2, x_3) = \frac{(1+\alpha_1)_n (1+\alpha_2)_n (1+\alpha_3)_n}{(n!)^2} {}_F^{(3)}\left(-n; -; -; 1+\alpha_3 + \beta_3 + n; 1+\alpha_2 + \beta_2 + n; 1+\alpha_1 + \beta_1 + n; -; -; -; 1+\alpha_3; 1+\alpha_2; 1+\alpha_1; \left(\frac{1-x_3}{2}\right)\left(\frac{1-x_2}{2}\right)\left(\frac{1-x_1}{2}\right)\right) \quad (1.13)$$

$$P_n^{(\alpha_1, \beta_1; \dots; \alpha_p, \beta_p)}(x_1, \dots, x_p) = \prod_{j=1}^p \frac{(1+\alpha_j)_n}{(n!)^p} {}_F^{(1; 1; \dots; 1)}\left(-n; 1+\alpha_p + \beta_p + n; \dots; 1+\alpha_1 + \beta_1 + n; 1+\alpha_p; \dots; 1+\alpha_1; \frac{1-x_p}{2}, \frac{1-x_{p-1}}{2}, \dots, \frac{1-x_1}{2}\right) \quad (1.14)$$

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Function F in R.H.S. of equation (1.12) (1.13) (1.14) represents Apples function for two variable, Hypergeometric function for three variables and Kample de fernet for multivariable.

$$\begin{aligned}
 H[Z_1, Z_2, \dots, Z_r] &= H_{A, B; \{A_i, B_i\}}^{O, N; \{M_i, N_i\}} \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} \begin{bmatrix} S; T \\ \vdots \\ S'; T' \end{bmatrix} \\
 &= H_{A, B; \{A_1, B_1; A_2, B_2; \dots; A_r, B_r\}}^{O, N; \{M_1, N_1; M_2, N_2; \dots; M_r, N_r\}} \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} \begin{bmatrix} (a_j; \alpha_j^{(i)}; \dots; \alpha_j^{(r)})_{1, A} (C_j^{(i)}, E_j^{(i)})_{1, A_r} \\ \vdots \\ (b_j; \beta_j^{(i)}; \dots; \beta_j^{(r)})_{1, B} (d_j^{(i)}, \delta_j^{(i)})_{1, B_r} \end{bmatrix} \\
 &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \theta(S_1 \dots S_r) \prod_{i=1}^r \left\{ \phi_i(S_i) Z_i^{S_i} dS_i \right\} \quad (1.21)
 \end{aligned}$$

where  $\omega = \sqrt{-1}$

$$S = (a_j, \alpha_j^{(i)} \dots \alpha_j^{(r)})_{1, A} \quad (1.22)$$

$$T = (C_j^{(i)}, E_j^{(i)})_{1, A_r} \quad (1.24)$$

$$S' = (b_j, \beta_j^{(i)} \dots \beta_j^{(r)})_{1, B} \quad (1.25)$$

$$T' = (d_j^{(i)}, \delta_j^{(i)})_{1, B_r} \quad (1.26)$$

$$\begin{aligned}
 \theta(S_1 \dots S_r) &= \frac{\prod_{j=1}^N \Gamma\left(1 - a_j + \sum_{i=1}^r \alpha_j S_i\right)}{\prod_{j=N+1}^A \Gamma\left(\alpha_j - \sum_{i=1}^r \alpha_i^{(i)} S_i\right) \prod_{j=1}^B \Gamma\left(1 - b_j + \sum_{i=1}^r \beta_j S_i\right)} \\
 \phi_i(S_i) &= \frac{\prod_{j=1}^{M_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} S_i) \prod_{j=1}^{N_i} \Gamma(1 - C_j^{(i)} + E_j^{(i)} S_i)}{\prod_{j=M_i+1}^{B_i} \Gamma(1 - d_j^{(i)} - \delta_j^{(i)} S_i) \prod_{j=N+1}^{A_i} \Gamma(C_j^{(i)} - E_j^{(i)} S_i)} \quad (1.27)
 \end{aligned}$$

i denotes the no of dashes for example  $b_i^{(i)} = b'$ ,  $b_i^{(2)} = b''$  ...etc.

For the expansion of multivariable H function and restriction of parameters see the research paper of Saxena, Shrivastava and Panda [ 3, pg. 271, equ.(4.1)]. In this research paper we will use the following result [4, pg 58]

$$\int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\alpha_p} (1+x_p)^{\sigma_p} \right] P_n^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)} (x_1 \dots x_r) dx_1 dx_2 \dots dx_r$$

$$\begin{aligned}
 &= \frac{1}{(n!)^r} \prod_{p=1}^r \left[ \frac{(-1)^{np} 2^{\sigma_p + \alpha_p + \frac{1}{r}} \Gamma(\alpha_p + n + 1) \Gamma(\sigma_p - \beta_p + 1) \Gamma(\sigma_p + 1)}{\Gamma(\sigma_p - \beta_p - n + 1) \Gamma(\sigma_p + \alpha_p + n + 2)} \right] \\
 &\quad {}_3F_2 \left[ \begin{matrix} \prod_{p=1}^r (-\lambda_p); \prod_{p=1}^r (\sigma_p - \beta_p + 1); \prod_{p=1}^r (\sigma_p + 1) \\ \prod_{p=1}^r (\sigma_p - \beta_p - n + 1); \prod_{p=1}^r (\sigma_p + \alpha_p + n + 2) \end{matrix}; 2^r \right] \\
 &\quad \int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\rho_p} (1+x_p)^{\beta_p} \right] P_n^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) dx_1 dx_2 \dots dx_r \tag{1.28}
 \end{aligned}$$

Third Integral

$$\begin{aligned}
 &\int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \left\{ \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\rho_p} (1+x_p)^{\beta_p} \right] \right\} \\
 &\quad P_n^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) H \left[ Z_1 x_1^{\mu_1} (1-x_1)^{\delta_1} \dots Z_r x_r^{\mu_r} (1-x_r)^{\delta_r} \right] dx_1 dx_2 \dots dx_r
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(n!)^r} \prod_{p=1}^r \left[ \frac{(-1)^{np} 2^{\rho_p + \beta_p + \frac{1}{r}} \Gamma(\beta_p + n + 1) \Gamma(\rho_p - \beta_p + 1) \Gamma(\rho_p + 1)}{\Gamma(\rho_p - \alpha_p - n + 1) \Gamma(\rho_p + \beta_p + n + 2)} \right] \\
 &\quad {}_3F_2 \left[ \begin{matrix} \prod_{p=1}^r (-\lambda_p); \prod_{p=1}^r (\rho_p - \beta_p + 1); \prod_{p=1}^r (\rho_p + 1) \\ \prod_{p=1}^r (\rho_p - \alpha_p - n + 1); \prod_{p=1}^r (\sigma_p + \beta_p + n + 2) \end{matrix}; 2^r \right] \\
 &\quad \int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\alpha_p} (1+x_p)^{\sigma_p} \right] H \left[ Z_1 \left( \frac{1+x_1}{x_1} \right)^{\mu_1} \dots Z_r \left( \frac{1+x_r}{x_r} \right)^{\mu_r} \right] dx_1 dx_2 \dots dx_r \tag{1.29}
 \end{aligned}$$

2. In this section we evaluate multivariable H function consisting four product of Jacobi function

We will use this integral in the expansion of  $H [Z_1, \dots, Z_r]$

$$\begin{aligned}
 &\int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \left\{ \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\alpha_p} (1+x_p)^{\sigma_p} \right] \right\} \\
 &\quad P_n^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) H \left[ Z_1 \left( \frac{1+x_1}{x_1} \right)^{\mu_1} \dots Z_r \left( \frac{1+x_r}{x_r} \right)^{\mu_r} \right] dx_1 dx_2 \dots dx_r \\
 &= \left\{ \prod_{p=1}^r \left[ (-1)^{np} 2^{\sigma_p + \alpha_p + \frac{1}{r}} \Gamma(\alpha_p + n + 1) \left( \sum_{k_p=0}^{\infty} \frac{2^{K_p}}{(k_p)!} \right) \right] \right\} H_{A, B; A_1+3, B_1+3; \dots; A_r+3, B_r+3}^{O, N; m_1, n_1+3; m_2, n_2+3; \dots; m_r, n_r+3} \\
 &\quad \begin{bmatrix} Z_1 & 2^{\mu_1} & \left| \begin{array}{c} \prod_{p=1}^r (\lambda_p + 1 - k_p; \mu_p), \prod_{p=1}^r (\beta_p - \sigma_p - k_p; \mu_p), \prod_{p=1}^r (-\sigma_p - k_p; \mu_p) \\ \dots \dots \dots \dots \dots \dots \dots \end{array} \right. \\ \cdot & \cdot & \left. \begin{array}{c} \prod_{p=1}^r (\lambda_p + 1; \mu_p), \prod_{p=1}^r (\beta_p - \sigma_p + n - k_p; \mu_p), \prod_{p=1}^r (-1 - \sigma_p - \alpha_p - n - k_p; \mu_p) \end{array} \right. \\ Z_r & 2^{\mu_r} & \end{bmatrix} \tag{2.1}
 \end{aligned}$$

Integral (2.1) is valid under the restriction of

$$\operatorname{Re}(\lambda_i) > -1,$$

$$\operatorname{Re}(\alpha_i) > -1,$$

$$|\arg Z_i| > \frac{\Omega\pi_i}{2},$$

$$\operatorname{Re}(\sigma_i + \mu_j \xi_j) > 0,$$

$$\operatorname{Re}(\lambda_i - \mu_j \xi_j) > 0$$

$$\text{where } \xi_j = \sum_{j=1}^r \min_{1 \leq m_r} \left( \operatorname{Re} \left( \frac{d_j^r}{\delta_j^r} \right) \right) \quad \forall i, j = 1 \dots r$$

where all  $\mu_1, \mu_2, \dots, \mu_r$  are positive and are not equal to zero.

Second integral

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \left\{ \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\alpha_p} (1+x_p)^{\sigma_p} \right] \right\} \\ & P_n^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) H \left[ Z_1 x_r (1+x_1)^{\mu_1} \dots Z_r x_r (1+x_r)^{\mu_r} \right] dx_1 dx_2 \dots dx_r \\ & = \frac{1}{(n!)^r} \left\{ \prod_{p=1}^r \left[ (-1)^{np} 2^{\sigma_p + \alpha_p + \frac{1}{r}} (\alpha_p + n + 1) \left( \sum_{k_p=0}^{\infty} \frac{2^{K_p}}{(k_p)!} \right) \right] \right\} H_{A, B; A_1+3, B_1+3; \dots; A_r+3, B_r+3}^{O, N; m_1, n_1+3; m_2, n_2+3; \dots; m_r, n_r+3} \\ & \left[ \begin{array}{c|c} Z_1 & 2^{\delta_1} \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r & 2^{\delta_r} \end{array} \middle| \begin{array}{c} \prod_{p=1}^r (-\lambda_p + k_p; \delta_p), \prod_{p=1}^r (\beta_p - \sigma_p - k_p; \delta_p), \prod_{p=1}^r (-\sigma_p - k_p; \delta_p) \\ \prod_{p=1}^r (\beta_p + n - \sigma_p - k_p; \delta_p), \prod_{p=1}^r (-\lambda_p; \mu_p) \prod_{p=1}^r (-\sigma_p - \alpha_p - n - k_p - 1; \delta_p) \end{array} \right] \end{aligned} \quad (2.2)$$

Integral (2.2) is valid under the restriction of

$$\operatorname{Re}(\alpha_i) > 1,$$

$$|\arg Z_i| > \frac{\Omega\pi_i}{2},$$

$$\operatorname{Re}(\lambda_i - \mu_j \xi_j) > 0,$$

$$\operatorname{Re}(\sigma_i + \mu_j \xi_j) > 0,$$

$$\text{where } \xi_j = \sum_{j=1}^r \min_{1 \leq m_r} \left( \operatorname{Re} \left( \frac{d_j^r}{\delta_j^r} \right) \right) \quad \forall i, j = 1 \dots r$$

where all  $\delta_1, \delta_2, \dots, \delta_r$  are positive and are not equal to zero.

$$\begin{aligned}
 &= \frac{1}{(n!)^r} \left\{ \prod_{p=1}^r \left[ (-1)^{np} 2^{\rho_p + \beta_p + \frac{1}{r}} \Gamma(\beta_p + n + 1) \left( \sum_{k_p=0}^{\infty} \frac{2^{k_p}}{(k_p)!} \right) \right] \right\} H_{A,B;A_1+3,B_1+3;\dots;A_r+3,B_r+3}^{O,N;m_1,n_1+3;m_2,n_2+3;\dots;m_r,n_r+3} \\
 &\quad \left[ Z_1 2^{\delta_1} \left| \begin{array}{l} \prod_{p=1}^r (-\lambda_p + k_p; \mu_p), \prod_{p=1}^r (\beta_p - \sigma_p - k_p; \delta_p), \prod_{p=1}^r (-\rho_p - k_p; \delta_p) \\ \cdot \quad \cdot \\ \cdot \quad \cdot \end{array} \right. \right. \\
 &\quad \left. \left. Z_r 2^{\delta_r} \left| \begin{array}{l} \prod_{p=1}^r (-\lambda_p; \mu_p), \prod_{p=1}^r (-\rho_p + \alpha_p + n - k_p; \delta_p), \prod_{p=1}^r (-\rho_p - \beta_p - n - k_p - 1; \rho_p) \end{array} \right. \right] \right] \tag{2.3}
 \end{aligned}$$

Integral (2.3) is valid under the following restrictions of:

$$\operatorname{Re}(\lambda_i) > -1,$$

$$\operatorname{Re}(\beta_i) > -1,$$

$$|\arg Z_i| > \frac{\Omega\pi_i}{2}$$

$$\operatorname{Re}(\rho_i + \delta_j, \xi_j) > 0,$$

$$\operatorname{Re}(\lambda_i - \mu_j, \xi_j) > 0$$

$$\text{where } \xi_j = \sum_{j=1}^r \min_{1 \leq m_r} \left( \operatorname{Re} \left( \frac{d_j^r}{\delta_j^r} \right) \right) \forall i, j = 1 \dots r$$

where all  $\mu_1, \mu_2, \dots, \mu_r$  and  $\delta_1, \delta_2, \dots, \delta_r$  are positive and are not equal to zero.

Fourth Integral

$$\begin{aligned}
 &\int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \left\{ \prod_{p=1}^r \left[ (x_p)^{\lambda_p} (1-x_p)^{\rho_p} (1+x_p)^{\beta_p} \right] \right\} \\
 &P_n^{(\alpha_1, \beta_1, \dots, \alpha_r, \beta_r)}(x_1 \dots x_r) H \left[ Z_1 x_1^{\mu_1} \left( \frac{1-x_1}{x_1} \right)^{\delta_1} \dots Z_r x_r^{\mu_r} \left( \frac{1-x_r}{x_r} \right)^{\delta_r} \right] dx_1 dx_2 \dots dx_r \\
 &= \frac{1}{(n!)^r} \left\{ \prod_{p=1}^r \left[ (-1)^{np} 2^{\rho_p + \beta_p + \frac{1}{r}} \Gamma(\beta_p + n + 1) \left( \sum_{k_p=0}^{\infty} \frac{2^{k_p}}{(k_p)!} \right) \right] \right\} H_{A,B;A_1+3,B_1+3;\dots;A_r+3,B_r+3}^{O,N;m_1,n_1+3;m_2,n_2+3;\dots;m_r,n_r+3} \\
 &\quad \left[ Z_1 2^{\mu_1} \left| \begin{array}{l} \prod_{p=1}^r (\lambda_p - k_p; \mu_p), \prod_{p=1}^r (\beta_p - \rho_p - k_p; \mu_p), \prod_{p=1}^r (-\rho_p - k_p; \mu_p) \\ \cdot \quad \cdot \\ \cdot \quad \cdot \end{array} \right. \right. \\
 &\quad \left. \left. Z_r 2^{\mu_r} \left| \begin{array}{l} \prod_{p=1}^r (-\lambda_p - 1; \mu_p), \prod_{p=1}^r (-\rho_p + n + \alpha_p - k_p; \mu_p), \prod_{p=1}^r (-\rho_p - \beta_p - n - k_p - 1; \mu_p) \end{array} \right. \right] \right] \tag{2.4}
 \end{aligned}$$

Integral (2.4) is valid under the following restrictions of:

$$\operatorname{Re}(\lambda_i) > 1,$$

$$\operatorname{Re}(\beta_i) > -1,$$

$$|\arg Z_i| > \frac{\Omega\pi_i}{2}$$

$\operatorname{Re}(\rho_i + \mu_j \xi_j) > 0,$

$\operatorname{Re}(\lambda_i - \mu_j \xi_j) > 0$

$$\text{where } \xi_j = \sum_{j=1}^r \min_{1 \leq m_r} \left( \operatorname{Re} \left( \frac{d_j^r}{\delta_j^r} \right) \right) \quad \forall i, j = 1 \dots r$$

where all  $\mu_1, \mu_2, \dots, \mu_r$  are positive and are not equal to zero.

**Proof:** To establish (2.1), expressing the multivariable H function on L.H.S. as Mellin Barnes Contour Integral (1.21) and then change the order of integration. This change will be valid under the given restriction and this change will be absolutely convergent. We get the following result

$$\frac{1}{(2\pi\omega)^r} \int_{L_1} \int_{L_2} \dots \int_{L_r} \frac{\theta(S_1 \dots S_r)}{(n!)^r} \left\{ \prod_{i=1}^r [\phi(S_i) Z_i] dS_i \right\} \\ \left\{ \int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 \prod_{i=1}^r [(x_p)^{\lambda_p - \mu_p S_p} \cdot (1-x_p)^{\alpha_p} \cdot (1+x_p)^{\sigma_p + \mu_p S_p}] P_n^{\alpha_1, \beta_1; \dots; \alpha_r, \beta_r}(x_1 \dots x_r) dx_1 dx_2 \dots dx_r \right\}$$

Evaluate the inner integral with the help of (1.28) we get the following result

$$\frac{1}{(2\pi\omega)^r} \int_{L_1} \int_{L_2} \dots \int_{L_r} \frac{\theta(S_1 \dots S_r)}{(n!)^r} \left\{ \prod_{p=1}^r [\phi(S_p) Z_p^{S_p}] dS_p \right\} \\ \left\{ \prod_{p=1}^r \left[ \frac{(-1)^{n_p} 2^{\sigma_p + \mu_p S_p + \frac{1}{r}} \Gamma(\sigma_p + n + 1) \Gamma(\sigma_p + \mu_p S_p - \beta_p + 1) \Gamma(\sigma_p + \mu_p S_p + 1)}{\Gamma(\sigma_p + \mu_p S_p - \beta_p - n + 1) \Gamma(\sigma_p + \mu_p S_p + \alpha_p + n + 2)} \right] \right\} \\ {}_3F_2 \left[ \begin{matrix} \prod_{p=1}^r (-\lambda_p + \mu_p S_p); \prod_{p=1}^r (\sigma_p + \mu_p S_p - \beta_p + 1); \prod_{p=1}^r (\sigma_p + \mu_p S_p + 1); \\ \prod_{p=1}^r (\sigma_p + \mu_p S_p - \beta_p - n + 1); \prod_{p=1}^r (\sigma_p + \mu_p S_p + \alpha_p + n + 2); 2^r \end{matrix} \right]$$

Now we represent the hypergeometric function into series and addition and integration [4, p 176 (75)] which is valid under the given restriction (2.1). At last using the definition of multivariable H which is defined in (1.21) we get the result (2.1).

## EXPANSION

The objective of this section is to apply the result of (2.1) to (2.4) then to express it in form H [Z<sub>1</sub>... Z<sub>r</sub>] and established four expansion formulas

### First expansion

$$\prod_{p=1}^r [(x_p)^{\lambda_p} \cdot (1-x_p)^{\sigma_p}] H \left[ Z_1 \left( \frac{1+x_1}{x_1} \right)^{\mu_1} \dots Z_r \left( \frac{1+x_r}{x_r} \right)^{\mu_r} \right] \\ = \left\{ \prod_{p=1}^r \left[ 2^{\sigma_p} \left( \sum_{k_p=0}^{\infty} \frac{2^{k_p}}{(k_p)!} \right) \right] \sum_{l=0}^{\infty} (1+\alpha_p + \beta_p + l)(1+\alpha_p + \beta_p + 2l)(-1)^{l_p} \right\} \\ P_l^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) H_{A, B; A_1+3, B_1+3; \dots; A_r+3, B_r+3}^{O, N; m_1, n_1+3; m_2, n_2+3; \dots; m_r, n_r+3}$$

$$\begin{bmatrix} Z_1 & 2^{\delta_1} \left| \prod_{p=1}^r (1 + \lambda_p - k_p; \mu_p), \prod_{p=1}^r (-k_p - \alpha_p; \mu_p), \prod_{p=1}^r (\sigma_p + \beta_p - k_p; \mu_p) \right. \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r & 2^{\delta_r} \left| \prod_{p=1}^r (1 + \lambda_p; \mu_p), \prod_{p=1}^r (-1 - \sigma_p - k_p; \mu_p), \prod_{p=1}^r (1 - \sigma_p - k_p; \mu_p) \right. \end{bmatrix} \quad (3.1)$$

Expansion (3.1) which is valid under the given restriction of (2.1) is also satisfied.

### Second expansion

$$\begin{aligned} & \prod_{p=1}^r \left[ (x_p)^{\lambda_p} \cdot (1 + x_p)^{\sigma_p} \right] H \left[ Z_1 x_1^{\mu_1} (1 + x_1)^{\delta_1} \dots Z_r x_r^{\mu_r} (1 + x_r)^{\delta_r} \right] \\ &= \left\{ \prod_{p=1}^r \left[ 2^{\sigma_p} \left( \sum_{k_p=0}^{\infty} \frac{2^{k_p}}{(k_p)!} \right) (-1)^{l_p} \right] \sum_{l=0}^{\infty} (1 + \alpha_p + \beta_p + l)(1 + \alpha_p + \beta_p + 2l) \right\} \\ & P_l^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)} (x_1 \dots x_r) H_{A, B; A_1+3, B_1+3; \dots; A_r+3, B_r+3}^{O, N; m_1, n_1+3; m_2, n_2+3; \dots; m_r, n_r+3;} \\ & \begin{bmatrix} Z_1 & 2^{\delta_1} \left| \prod_{p=1}^r (-\lambda_p; \mu_p), \prod_{p=1}^r (-\sigma_p - k_p; \delta_p), \prod_{p=1}^r (-k_p - \sigma_p - \beta_p; \delta_p) \right. \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r & 2^{\delta_r} \left| \prod_{p=1}^r (k_p - \lambda_p; \mu_p), \prod_{p=1}^r (1 - \sigma_p - k_p; \delta_p), \prod_{p=1}^r (1 - l - 2 - \rho_p; \delta_p) \right. \end{bmatrix} \end{aligned} \quad (3.2)$$

Expansion (3.2) which is valid under the given restriction of (2.2) is also satisfied.

### Third expansion

$$\begin{aligned} & \prod_{p=1}^r \left[ (x_p)^{\lambda_p} \cdot (1 - x_p)^{\rho_p} \right] H \left[ Z_1 x_1^{\mu_1} (1 - x_1)^{\delta_1} \dots Z_r x_r^{\mu_r} (1 - x_r)^{\delta_r} \right] \\ &= \left\{ \prod_{p=1}^r \left[ 2^{\rho_p} \left( \sum_{k_p=0}^{\infty} \frac{2^{k_p}}{(k_p)!} \right) (-1)^{l_p} \right] \sum_{l=0}^{\infty} (1 + \alpha_p + \beta_p + l)(1 + \alpha_p + \beta_p + 2l) \right\} \\ & P_l^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)} (x_1 \dots x_r) H_{A, B; A_1+3, B_1+3; \dots; A_r+3, B_r+3}^{O, N; m_1, n_1+3; m_2, n_2+3; \dots; m_r, n_r+3;} \\ & \begin{bmatrix} Z_1 & 2^{\delta_1} \left| \prod_{p=1}^r (-\lambda_p; \mu_p), \prod_{p=1}^r (k_p - \rho_p; \delta_p), \prod_{p=1}^r (-k_p - \rho_p - l; \delta_p) \right. \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r & 2^{\delta_r} \left| \prod_{p=1}^r (k_p - \lambda_p; \mu_p), \prod_{p=1}^r (l - \rho_p - k_p; \delta_p), \prod_{p=1}^r (1 - \alpha_p - \beta_p - l - k_p - \rho_p; \delta_p) \right. \end{bmatrix} \end{aligned} \quad (3.3)$$

Expansion (3.3) which is valid under the given restriction of (2.3) is also satisfied.

#### Fourth expansion

$$\begin{aligned}
 & \prod_{p=1}^r \left[ (x_p)^{\lambda_p} \cdot (1-x_p)^{\rho_p} \right] \quad H \left[ Z_1 \left( \frac{1-x_1}{x_1} \right)^{\mu_1} \dots Z_r \left( \frac{1-x_r}{x_r} \right)^{\mu_r} \right] \\
 & = \left\{ \prod_{p=1}^r \left[ 2^{\rho_p} \left( \sum_{k_p=0}^{\infty} \frac{2^{k_p}}{(k_p)!} \right) (-1)^{l_p} \right] \sum_{l=0}^{\infty} (1+\alpha_p + \beta_p + l)(1+\alpha_p + \beta_p + 2l) \right\} \\
 & P_l^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) \quad H_{A, B; A_1+3, B_1+3; \dots; A_r+3, B_r+3}^{O, N; m_1, n_1+3; m_2, n_2+3; \dots; m_r, n_r+3;} \\
 & \left[ \begin{array}{c|c} Z_1 & 2^{\delta_1} \left[ \prod_{p=1}^r (1+\lambda_p - k_p; \mu_p), \prod_{p=1}^r (\rho_p - k_p; \mu_p), \prod_{p=1}^r (\rho_p + \alpha_p - k_p; \mu_p) \right] \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r & 2^{\delta_r} \left[ \prod_{p=1}^r (1+\lambda_p; \mu_p), \prod_{p=1}^r (1-k_p - \rho_p; \mu_p), \prod_{p=1}^r (1-\rho_p - k_p - \beta_p - l; \mu_p) \right] \end{array} \right] \quad (3.4)
 \end{aligned}$$

**Proof:** To establish (3.1) let

$$\prod_{p=1}^r \left[ (x_p)^{\lambda_p} \cdot (1-x_p)^{\rho_p} \right] \quad H \left[ Z_1 \left( \frac{1+x_1}{x_1} \right)^{\mu_1} \dots Z_r \left( \frac{1+x_r}{x_r} \right)^{\mu_r} \right] = \sum_{l=0}^{\infty} m_l P_l^{(\alpha_1, \beta_1; \dots; \alpha_r, \beta_r)}(x_1 \dots x_r) \quad (3.5)$$

where  $m_l$  is a constant. For finding this we multiply  $\prod_{p=1}^r \left[ (1-x_p)^{\alpha_p} \cdot (1+x_p)^{\beta_p} \right]$  on both sides of the equation (3.5)

and then integrate in the limit  $-1$  to  $+1$ . Lastly in the left side result (2.1) is recalculated and on right side orthogonal property of Jacobi polynomial is applied to get the desired result [5, pg 285(5) and (9)]

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