



IN GIVEN REGION THE NUMBER OF ZEROS OF POLYNOMIAL

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ABSTRACT

Polynomials in various forms have recently been worked out for their extensive application in stability problems, coding theory, signal processing, electrical networks, linear control systems etc. The results concerning the number of zeros in a region has been of great use. In this paper we prove some results concerning the bounds for the zeros of a polynomial, which generalizes the result due to Bidkam and Dewan and also results due to Gulshan Singh and W.M. Shah

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The following results concerning the number of zeros in a closed disk to Q G Mohammad [9]

Theorem A: If $P(z) = \sum_{r=0}^n a_r z^r$ is a polynomial of degree n , such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0.$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}$. theorem for different classes of polynomials and proved the following result:

Bidkam and Dewan [3] generalized the above

Theorem B: If $P(z) = \sum_{r=0}^n a_r z^r$ is a polynomial of degree n , such that

$$a_n \leq a_{n-1} \dots \leq a_{\lambda+1} \leq a_{\lambda} \geq a_{\lambda-1} \geq \dots \geq a_1 \geq a_0.$$

for some $\lambda, 0 \leq \lambda \leq n$, then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \left\{ \log \frac{|a_n| + |a_0| - a_n - a_0 + 2a_{\lambda}}{|a_0|} \right\}.$$

Recently Gulshan Singh and W. M. Shah extended [12] the above theorems to the polynomials with complex coefficient and proved some more results which generalized both Theorems A as well as Theorem B. The following results were proved.

Theorem C: Let $P(z) = \sum_{r=0}^n a_r z^r$ be a polynomial of degree n , such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j , $j = 0, 1, 2, \dots, n$ are real numbers. if for some positive numbers k_1, k_2, ρ_1 and ρ_2 with $k_1 \geq 1, k_2 \geq 1$ and $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1$.

$$k_1 \alpha_n \geq \alpha_{n-1} \geq \dots \alpha_1 \geq \rho_1 \alpha_0$$

$$k_2 \beta_n \geq \beta_{n-1} \geq \dots \beta_1 \geq \rho_2 \beta_0$$

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Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \left\{ \log \frac{B_1}{|a_0|} \right\}$$

$$B_1 = k_1(|\alpha_n| + \alpha_n) - \rho_1(|\alpha_0| + \alpha_0) + 2|\alpha_0| + k_2(|\beta_n| + \beta_n) - \rho_2(|\beta_0| + \beta_0) + 2|\beta_0|$$

The following theorem is generalization of Theorem B where instead of real coefficients, the complex coefficients has been considered

Theorem D: Let $P(z) = \sum_{r=0}^n a_r z^r$ be a polynomial of degree n , such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j , $j = 0, 1, 2, \dots, n$ are real numbers. if for some positive numbers k_1, k_2, ρ_1 and ρ_2 with $k_1 \geq 1, k_2 \geq 1$ and $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1$, and for some $0 \leq \lambda \leq n$.

$$k_1 \alpha_n \leq \alpha_{n-1} \leq \dots \leq \alpha_{\lambda+1} \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \dots \geq \alpha_1 \geq \rho_1 \alpha_0$$

$$k_2 \beta_n \leq \beta_{n-1} \leq \dots \leq \beta_{\lambda+1} \leq \beta_\lambda \geq \beta_{\lambda-1} \geq \dots \geq \beta_1 \geq \rho_2 \beta_0$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{R}{2}$ does not exceed

$$\frac{1}{\log 2} \left\{ \log \frac{M_1}{|a_0|} \right\} \quad \text{for } R \geq 1$$

and

$$\frac{1}{\log 2} \left\{ \log \frac{M_2}{|a_0|} \right\} \quad \text{for } R \leq 1$$

Where

$$M_1 = (|\alpha_n| + |\beta_n|)R^{n+1} + (|\alpha_n|(1 - k_1) + |\beta_n|(1 - k_2))R^n + (|\alpha_0|(1 - \rho_1) + |\beta_0|(1 - \rho_2))R + (|\alpha_0| + |\beta_0|) + R^\lambda(\alpha_\lambda + \beta_\lambda - \rho_1\alpha_0 - \rho_2\beta_0) + R^n(\alpha_\lambda + \beta_\lambda - k_1\alpha_n - k_2\beta_n)$$

and

$$M_2 = (|\alpha_n| + |\beta_n|)R^{n+1} + (|\alpha_n|(1 - k_1) + |\beta_n|(1 - k_2))R^n + (|\alpha_0|(1 - \rho_1) + |\beta_0|(1 - \rho_2))R + (|\alpha_0| + |\beta_0|) + R(\alpha_\lambda + \beta_\lambda - \rho_1\alpha_0 - \rho_2\beta_0) + R^\lambda(\alpha_\lambda + \beta_\lambda - k_1\alpha_n - k_2\beta_n)$$

In this paper we will generalize Theorem A, Band Theorem C will also be generalized as a Corollary of the theorem which states as

Theorem 1: If $P(z) = \sum_{r=0}^n a_r z^r$ is a polynomial of degree n , such that if for some numbers λ and, where $\lambda > 0$ and $\mu < 0$.

$$\lambda + a_n \geq a_{n-1} \geq \dots \geq a_1 \geq \mu + a_0.$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed $\frac{1}{\log 2} \log \frac{M}{|a_0|}$,

where

$$M = (|a_n| + a_n) + (|a_0| + a_0) + 2\lambda + 2\mu$$

For the proof of these theorem we need following lemmas

Lemma 1.: Assume that f is analytic in a disk $|z| \leq R$, but not identically zero. Assume also that $f(0) \neq 0$. Let f have zeros $\{a_k\}, k = 1, 2, \dots, n$ in $|z| \leq R$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)| = \sum_{j=1}^n \log \frac{R}{|a_j|}$$

The above lemma is well known Jensen 's Theorem. The following lemma can be easily deduced from lemma 1

Lemma 2: If $f(z)$ is regular, $f(0) \neq 0$. and $|f(z)| \leq M(r)$ in $|z| \leq r$, then the number of zeros of $f(z)$ in $|z| \leq \frac{r}{2}$ does not exceed

$$\frac{1}{\log 2} \left\{ \log \frac{M(r)}{|f(0)|} \right\}$$

Remark 1: If we put $\lambda = \mu = 0$ we obtain Theorem A

Remark 2: If we put $\lambda = (k_1 - 1)a_n$ and $\mu = (\rho_1 - 1)a_0$ the above theorem reduces to Theorem C for real coefficients of polynomial.

Proof of the theorem 1:

Consider the polynomial

$$Q(z) = (1 - z)P(z)$$

$$Q(z) = -a_n z^{n+1} - \lambda z^n + (\lambda + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_1 - \mu - a_0)z + \mu z + a_0$$

$$Q(z) = -a_n z^{n+1} - \lambda z^n + \mu z + a_0 + (a_1 - \mu - a_0)z + (a_2 - a_1)z^2 + \dots + (a_{n-1} - a_{n-2})z^{n-1} + (\lambda + a_n - a_{n-1})z^n$$

$$|Q(z)| \leq |a_n||z|^{n+1} + |\lambda||z|^n + |a_0| + |\mu||z| + |a_1 - \mu - a_0||z| + |a_2 - a_1||z|^2 + \dots + |a_{n-1} - a_{n-2}||z|^{n-1} + |\lambda + a_n - a_{n-1}||z|^n$$

For $|z| \leq 1$

$$\begin{aligned} |Q(z)| &\leq |a_n| + |\lambda| + |a_0| + |\mu| + |a_1 - \mu - a_0| + |a_2 - a_1| + \dots + |a_{n-1} - a_{n-2}| + |\lambda + a_n - a_{n-1}| \\ &= |a_n| + |\lambda| + |a_0| + |\mu| + a_1 - \mu - a_0 + a_2 - a_1 + \dots + a_{n-1} - a_{n-2} + \lambda + a_n - a_{n-1} \\ &= |a_n| + \lambda + |a_0| + |\mu| + -\mu - a_0 + \lambda + a_n \\ &= (|a_n| + a_n) + (|a_0| - a_0) + 2\lambda + 2|\mu| \\ &= M \end{aligned}$$

Thus $|Q(z)| \leq M$

Also $Q(0) = a_0$, therefore by applying lemma 2 to $Q(z)$, we get the number of zeros of $Q(z)$ in $|z| \leq 1/2$ does not exceed $\frac{1}{\log 2} \log \frac{M}{|a_0|}$.

where

$$M = (|a_n| + a_n) + (|a_0| - a_0) + 2\lambda + 2\mu$$

Since the number of zeros of $P(z)$ in $|z| \leq 1/2$ does not exceed the number of zeros of $Q(z)$ in $|z| \leq 1/2$ the proof of the theorem is complete.

Theorem 2: If $P(z) = \sum_{r=0}^n a_r z^r$ is a polynomial of degree n , such that if for some α and β where α and β are both negative numbers and for some λ , $0 \leq \lambda \leq n$

$$\alpha + a_n \leq a_{n-1} \leq \dots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \dots \geq a_1 \geq \beta + a_0$$

Then all the zeros of $P(z)$ in $|z| \leq R/2$, ($R > 0$) does not exceed

$$\frac{1}{\log 2} \log \frac{M_1}{|a_0|} \quad \text{for } R \geq 1$$

and

$$\frac{1}{\log 2} \log \frac{M_2}{|a_0|} \quad \text{for } R < 1$$

where

$$\begin{aligned} M_1 &= |a_n|R^{n+1} + |\alpha|R^n + |\beta|R + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha - \beta\} \text{ and} \\ M_2 &= |a_n|R^{n+1} + |\alpha|R^n + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha + 2|\beta|\} \end{aligned}$$

Remark 3: If we put $\alpha = \beta = 0$ we obtain Theorem B

Remark 4: if we put $\alpha = (k_1 - 1)a_n$ and $\beta = (\rho_1 - 1)a_0$ the above theorem reduces to Theorem D for real coefficients of polynomial.

Proof of the theorem 2:

$$Q(z) = (1 - z)P(z)$$

$$Q(z) = -a_n z^{n+1} - \alpha z^n + \beta z + a_0 + (\alpha + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + \dots + (a_2 - a_1)z^2 + (a_1 - \beta - a_0)z$$

$$|Q(z)| \leq |a_n||z|^{n+1} + |\alpha||z|^n + |\mu||z| + |a_0| + |\alpha + a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \dots + |a_{\lambda+1} - a_\lambda||z|^{\lambda+1} + |a_\lambda - a_{\lambda-1}||z|^\lambda + |a_{\lambda-1} - a_{\lambda-2}||z|^{\lambda-1} + \dots + |a_2 - a_1||z|^2 + |a_1 - \beta - a_0|z$$

Now for $|z| \leq R$

$$|Q(z)| \leq |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + |\alpha + a_n - a_{n-1}|R^n + |a_{n-1} - a_{n-2}|R^{n-1} + \dots + |a_{\lambda+1} - a_\lambda|R^{\lambda+1} + |a_\lambda - a_{\lambda-1}|R^\lambda + |a_{\lambda-1} - a_{\lambda-2}|R^{\lambda-1} + \dots + |a_2 - a_1|R^2 + |a_1 - \beta - a_0|R$$

Now we consider two cases for R

Case - 1: when $R \geq 1$ then we note that

$$R^n \geq R^{n-1} \geq R^{n-2} \geq \dots \geq R$$

Therefore

$$|Q(z)| \leq |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R^n\{a_{n-1} - a_n - \alpha + a_{n-2} - a_{n-1} + \dots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + a_{\lambda-1} - a_{\lambda-2} + a_{\lambda-1} - a_{\lambda-2} + \dots + a_2 - a_1 + a_1 - \beta - a_0\}$$

$$|Q(z)| \leq |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R^n\{2a_\lambda - a_0 - a_n - \alpha - \beta\} = M_1$$

Case - 2: when $R < 1$ then we note that

$$R \geq R^2 \geq R^3 \geq \dots \geq R^n$$

Therefore

$$|Q(z)| \leq |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R\{a_{n-1} - a_n - \alpha + a_{n-2} - a_{n-1} + \dots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + a_{\lambda-1} - a_{\lambda-2} + a_{\lambda-1} - a_{\lambda-2} + \dots + a_2 - a_1 + a_1 - \beta - a_0\}$$

$$|Q(z)| \leq |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R\{2a_\lambda - a_0 - a_n - \alpha + |\beta|\} = M_2$$

Also $Q(0) = a_0$, therefore by applying lemma 2 to $Q(z)$, we get the number of zeros of $Q(z)$ in $|z| \leq R/2$ does not exceed

$$\frac{1}{\log 2} \log \frac{M_1}{|a_0|} \quad \text{for } R \geq 1$$

and

$$\frac{1}{\log 2} \log \frac{M_2}{|a_0|} \quad \text{for } R < 1$$

where

$$M_1 = |a_n|R^{n+1} + |\alpha|R^n + |\beta|R + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha - \beta\} \text{ and}$$

$$M_2 = |a_n|R^{n+1} + |\alpha|R^n + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha + 2|\beta|\}$$

Since the number of zeros of $P(z)$ in $|z| \leq R/2$ does not exceed the number of zeros of $Q(z)$ in $|z| \leq R/2$ the proof of the theorem is complete.

Corollary 1: Let $P(z) = \sum_{r=0}^n a_r z^r$ be a polynomial of degree n , such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j , $j = 0, 1, 2, \dots, n$ are real numbers. if for some numbers $\alpha_1, \alpha_2, \beta_1$ and β_2 are all negative numbers and for some $0 \leq \lambda \leq n$.

$$\alpha_1 + \alpha_n \leq \alpha_{n-1} \leq \dots \leq \alpha_{\lambda+1} \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \dots \alpha_1 \geq \beta_1 + \alpha_0$$

$$\alpha_2 + \beta_n \leq \beta_{n-1} \leq \dots \leq \beta_{\lambda+1} \leq \beta_\lambda \geq \beta_{\lambda-1} \geq \dots \geq \beta_1 \geq \beta_2 + \beta_0$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{R}{2}$ does not exceed

$$\frac{1}{\log 2} \left\{ \log \frac{M_1}{|a_0|} \right\} \quad \text{for } R \geq 1$$

and

$$\frac{1}{\log 2} \left\{ \log \frac{M_2}{|a_0|} \right\} \quad \text{for } R \leq 1$$

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