International Research Journal of Pure Algebra -4(7), 2014, 566-570
Available online through www.rjpa.info ISSN 2248–9037

IN GIVEN REGION THE NUMBER OF ZEROS OF POLYNOMIAL

M. A. Kawoosa*

Govt. P. G., A. S. College – 190002, Srinager, Kashmir, India.

(Received on: 08-06-14; Revised & Accepted on: 12-07-14)

ABSTRACT

Polynomials in various forms have recently been worked out for their extensive application in stability problems, coding theory, signal processing, electrical networks, linear control systems etc. The results concerning the number of zeros in a region has been of great use. In this paper we prove some results concerning the bounds for the zeros of a polynomial, which generalizes the result due to Bidkam and Dewan and also results due to Gulshan Singh and W.M. Shah

Keywords: *Polynomials, zeros, generalizations.*

Mathematics Subject Classification (2000): *26C10, 12D10.*

The following results concerning the number of zeros in a closed disk to Q G Mohammad [9]

Theorem A: lf $P(z) = \sum_{r=0}^{n} a_r z^r$ is a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$1 + \dfrac{1}{log2} log \dfrac{a_n}{a_0}$. theorem for different classes of polynomials and proved the following result:

Bidkam and Dewan [3] generalized the above

Theorem B: lf $P(z) = \sum_{r=0}^{n} a_r z^r$ is a polynomial of degree n, such that

$$a_n \leq a_{n-1} \ldots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0.$$

for someλ, $0 \leq \lambda \leq n$, then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{log2}\left\{log \frac{\left||a_n| + |a_0| - a_n - a_0 + 2a_\lambda\right|}{|a_0|}\right\}.$$

Recently Gulshan Singh and W. M. Shah extended [12] the above theorems to the polynomials with complex co-efficient and proved some more results which generalized both Theorems A as well as Theorem B. The following results were proved.

Theorem C: Let $P(z) = \sum_{r=0}^{n} a_r z^r$ be a polynomial of degree n, such that $a_j = \alpha_j + i\beta_j,$ where α_j and $\beta_j,$ $j = 01,2,\ldots,n$ are real numbers. if for some positive numbers $k_1, k_2, \rho_1 and\ \rho_2$ with $k_1 \geq 1, k_2 \geq 1$ and $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1$.

$$k_1\alpha_n \geq \alpha_{n-1} \geq \cdots \alpha_1 \geq \rho_1\alpha_0$$

$$k_2\beta_n \geq \beta_{n-1} \geq \cdots \beta_1 \geq \rho_2\beta_0$$

Corresponding author: M. A. Kawoosa
Govt. P. G., A. S. College – 190002, Srinager, Kashmir, India.

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{log2}\left\{log\frac{B_1}{|a_0|}\right\}$$

$$B_1 = k_1(|\alpha_n| + \alpha_n) - \rho_1(|\alpha_0| + \alpha_0) + 2|\alpha_0| + k_2(|\beta_n| + \beta_n) - \rho_2(|\beta_0| + \beta_0) + 2|\beta_0|$$

The following theorem is generalization of Theorem B where instead of real coefficients, the complex coefficients has been considered

Theorem D: Let $P(z) = \sum_{r=0}^{n} a_r z^r$ be a polynomial of degree n, such that $a_j = \alpha_j + i\beta_{j,}$ where α_j and $\beta_{j,}$ $j = 01,2, \dots, n$ are real numbers. if for some positive numbers k_1, k_2, ρ_1 and ρ_2 with $k_1 \geq 1$, $k_2 \geq 1$ and $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1$, and for some $0 \leq \lambda \leq n$.

$$k_1\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{\lambda+1} \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \alpha_1 \geq \rho_1\alpha_0$$

$$k_2\beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_{\lambda+1} \leq \beta_\lambda \geq \beta_{\lambda-1} \geq \cdots \geq \beta_1 \geq \rho_2\beta_0$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{R}{2}$ does not exceed

$$\frac{1}{log2}\left\{log\frac{M_1}{|a_0|}\right\} \qquad for\ R \geq 1$$

and

$$\frac{1}{log2}\left\{log\frac{M_2}{|a_0|}\right\} \qquad for\ R \leq 1$$

Where

$$M_1 = (|\alpha_n| + |\beta_n|)R^{n+1} + \big(|\alpha_n|(1-k_1)\big) + |\beta_n|(1-k_2)R^n + \big(|\alpha_0|(1-\rho_1) + |\beta_0|(1-\rho_2)\big)R + (|\alpha_0| + |\beta_0|)$$
$$+ R^\lambda(\alpha_\lambda + \beta_\lambda - \rho_1\alpha_0 - \rho_2\beta_0) + R^n(\alpha_\lambda + \beta_\lambda - k_1\alpha_n - k_2\beta_n)$$

and

$$M_2 = (|\alpha_n| + |\beta_n|)R^{n+1} + \big(|\alpha_n|(1-k_1)\big) + |\beta_n|(1-k_2)R^n + \big(|\alpha_0|(1-\rho_1) + |\beta_0|(1-\rho_2)\big)R + (|\alpha_0| + |\beta_0|)$$
$$+ R(\alpha_\lambda + \beta_\lambda - \rho_1\alpha_0 - \rho_2\beta_0) + R^\lambda(\alpha_\lambda + \beta_\lambda - k_1\alpha_n - k_2\beta_n)$$

In this paper we will generalize Theorem A, Band Theorem C will also be generalized as a Corollary of the theorem which states as

Theorem 1: If $P(z) = \sum_{r=0}^{n} a_r z^r$ is a polynomial of degree n, such that if for some numbers λ and, where $\lambda > 0$ and $\mu < 0$.

$$\lambda + a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq \mu + a_0.$$

Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed $\frac{1}{log\,2}log\frac{M}{|a_0|}$.

where

$$M = (|a_n| + a_n) + (|a_0| + a_0) + 2\lambda + 2\mu$$

For the proof of these theorem we need following lemmas

Lemma 1:. Assume that f is analytic in a disk $|z| \leq R$, but not identically zero. Assume also that $f(0) \neq 0$. Let f have zeros $\{a_k\}, k = 1,2, \dots n$ in $|z| \leq R$, then

$$\frac{1}{2\pi}\int_0^{2\pi} log\left|f\left(Re^{i\vartheta}\right)\right|d\vartheta - log|f(0)| = \sum_{j=1}^{n} log\frac{R}{|a_j|}$$

The above lemma is well known Jensen ‘s Theorem. The following lemma can be easily deduced from lemma 1

Lemma 2: If $f(z)$ is regular, $f(0) \neq 0$. and $|f(z)| \leq M(r)$ in $|z| \leq r$, then the number of zeros of $f(z)$ in$|z| \leq \frac{r}{2}$ does not exceed

$$\frac{1}{log2}\left\{log\frac{M(r)}{f(0)}\right\}$$

Remark 1: If we put $\lambda = \mu = 0$ we obtain Theorem A

Remark 2: If we put $\lambda = (k_1 - 1)a_n$ and $\mu = (\rho_1 - 1)a_0$ the above theorem reduces to Theorem C for real coefficients of polynomial.

Proof of the theorem 1:

Consider the polynomial

$Q(z) = (1 - z)P(z)$

$Q(z) = -a_n z^{n+1} - \lambda z^z + (\lambda + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - \mu - a_0)z + \mu z + a_0$

$$Q(z) = -a_n z^{n+1} - \lambda z^z + +\mu z + a_0 + (a_1 - \mu - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_{n-1} - a_{n-2})z^{n-1} + (\lambda + a_n - a_{n-1})z^n$$

$$|Q(z)| \leq |a_n||z|^{n+1} + |\lambda||z|^n + |a_0| + |\mu||z| + |a_1 - \mu - a_0||z| + |a_2 - a_1||z|^2 + \cdots + |a_{n-1} - a_{n-2}||z|^{n-1} + |\lambda + a_n - a_{n-1}||z|^n$$

For $|z| \leq 1$

$$|Q(z)| \leq |a_n| + |\lambda| + |a_0| + |\mu| + |a_1 - \mu - a_0| + |a_2 - a_1| + \cdots + |a_{n-1} - a_{n-2}| + |\lambda + a_n - a_{n-1}|$$

$$= |a_n| + |\lambda| + |a_0| + |\mu| + a_1 - \mu - a_0 + a_2 - a_1 + \cdots + a_{n-1} - a_{n-2} + \lambda + a_n - a_{n-1}$$

$$= |a_n| + \lambda + |a_0| + |\mu| + -\mu - a_0 + \lambda + a_n$$

$$= (|a_n| + a_n) + (|a_0| - a_0) + 2\lambda + 2|\mu|$$

$$= M$$

Thus $|Q(z)| \leq M$

Also $Q(0) = a_0$, therefore by applying lemma 2 to $Q(z)$, we get the number of zeros of $Q(z)$ in $|z| \leq 1/2$ does not exceed $\frac{1}{\log 2} \log \frac{M}{|a_0|}$.

where

$$M = (|a_n| + a_n) + (|a_0| - a_0) + 2\lambda + 2\mu$$

Since the number of zeros of $P(z)$ in $|z| \leq 1/2$ does not exceed the number of zeros of $Q(z)$ in $|z| \leq 1/2$ the proof of the theorem is complete.

Theorem 2: lf $P(z) = \sum_{r=0}^{n} a_r z^r$ is a polynomial of degree n, such that if for some α and β where α and β are both negative numbers and for some λ, $0 \leq \lambda \leq n$

$$\alpha + a_n \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq \beta + a_0$$

Then all the zeros of $P(z)$ in $|z| \leq R/2$, $(R > 0)$ does not exceed

$$\frac{1}{\log 2} \log \frac{M_1}{|a_0|} \qquad \text{for } R \geq 1$$

and

$$\frac{1}{\log 2} \log \frac{M_2}{|a_0|} \qquad \text{for} R < 1$$

where

$M_1 = |a_n|R^{n+1} + |\alpha|R^n + |\beta|R + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha - \beta\}$ and
$M_2 = |a_n|R^{n+1} + |\alpha|R^n + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha + 2|\beta|\}$

Remark 3: If we put $\alpha = \beta = 0$ we obtain Theorem B

Remark 4: if we put $\alpha = (k_1 - 1)a_n$ and $\beta = (\rho_1 - 1)a_0$ the above theorem reduces to Theorem D for real coefficients of polynomial.

Proof of the theorem 2:

$Q(z) = (1-z)P(z)$

$$Q(z) = -a_n z^{n+1} - \alpha z^n + \beta z + a_0 + (\alpha + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + \cdots + (a_2 - a_1)z^2 + (a_1 - \beta - a_0)z$$

$$|Q(z)| \le |a_n||z|^{n+1} + |\alpha||z|^n + |\mu||z| + |a_0| + |\alpha + a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{\lambda+1} - a_\lambda||z|^{\lambda+1} + |a_\lambda - a_{\lambda-1}||z|^\lambda + |a_{\lambda-1} - a_{\lambda-2}||z|^{\lambda-1} + \cdots + |a_2 - a_1||z|^2 + |a_1 - \beta - a_0|z$$

Now for $|z| \le R$

$$|Q(z)| \le |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + |\alpha + a_n - a_{n-1}|R^n + |a_{n-1} - a_{n-2}|R^{n-1} + \cdots + |a_{\lambda+1} - a_\lambda|R^{\lambda+1} + |a_\lambda - a_{\lambda-1}|R^\lambda + |a_{\lambda-1} - a_{\lambda-2}|R^{\lambda-1} + \cdots + |a_2 - a_1|R^2 + |a_1 - \beta - a_0|R$$

Now we consider two cases for R

Case - 1: when $R \ge 1$ then we note that

$$R^n \ge R^{n-1} \ge R^{n-2} \ge \cdots \ge R$$

Therefore

$$|Q(z)| \le |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R^n\{a_{n-1} - a_n - \alpha + a_{n-2} - a_{n-1} + \cdots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + a_{\lambda-1} - a_{\lambda-2} + a_{\lambda-1} - a_{\lambda-2} + \cdots + a_2 - a_1 + a_1 - \beta - a_0\}$$

$$|Q(z)| \le |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R^n\{2a_\lambda - a_0 - a_n - \alpha - \beta\} = M_1$$

Case - 2: when $R < 1$ then we note that

$$R \ge R^2 \ge R^3 \ge \cdots \ge R^n$$

Therefore

$$|Q(z)| \le |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R\{a_{n-1} - a_n - \alpha + a_{n-2} - a_{n-1} + \cdots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + a_{\lambda-1} - a_{\lambda-2} + a_{\lambda-1} - a_{\lambda-2} + \cdots + a_2 - a_1 + a_1 - \beta - a_0\}$$

$$|Q(z)| \le |a_n|R^{n+1} + |\alpha|R^n + |\mu|R + |a_0| + R\{2a_\lambda - a_0 - a_n - \alpha + |\beta|\} = M_2$$

Also $Q(0) = a_0$, therefore by applying lemma 2 to $Q(z)$, we get the number of zeros of $Q(z)$ in $|z| \le R/2$ does not exceed

$$\frac{1}{log\,2} log \frac{M_1}{|a_0|} \qquad \text{for } R \ge 1$$

and

$$\frac{1}{log\,2} log \frac{M_2}{|a_0|} \qquad \text{for} R < 1$$

where

$M_1 = |a_n|R^{n+1} + |\alpha|R^n + |\beta|R + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha - \beta\}$ and
$M_2 = |a_n|R^{n+1} + |\alpha|R^n + |a_0| + R^n\{2a_\lambda - a_n - a_0 - \alpha + 2|\beta|\}$

Since the number of zeros of $P(z)$ in $|z| \le R/2$ does not exceed the number of zeros of $Q(z)$ in $|z| \le R/2$ the proof of the theorem is complete.

Corollary 1: Let $P(z) = \sum_{r=0}^{n} a_r z^r$ be a polynomial of degree n ,such that $a_j = \alpha_j + i\beta_j$, where α_j and β_j, $j = 01,2,\dots,n$ are real numbers. if for some numbers $\alpha_1, \alpha_2, \beta_1$ *and* β_2 are all negative numbers and for some $0 \le \lambda \le n$.

$$\alpha_1 + \alpha_n \le \alpha_{n-1} \le \cdots \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \cdots \alpha_1 \ge \beta_1 + \alpha_0$$

$$\alpha_2 + \beta_n \le \beta_{n-1} \le \cdots \le \beta_{\lambda+1} \le \beta_\lambda \ge \beta_{\lambda-1} \ge \cdots \ge \beta_1 \ge \beta_2 + \beta_0$$

Then the number of zeros of $P(z)$ in $|z| \le \frac{R}{2}$ does not exceed

$$\frac{1}{log2}\left\{log\frac{M_1}{|a_0|}\right\} \qquad for\ R \ge 1$$

and

$$\frac{1}{log2}\left\{log\frac{M_2}{|a_0|}\right\} \qquad for\ R \le 1$$

REFERENCES

[1] Lars V. Ahlfors, Complex Analysis, 3rd ed., McGraw. Hill.

[2] H. Alzer, Bounds for the Zeros of polynomials Complex Variable, Theory Appl., 8(2001), 143-150.

[3] M. Bidkham and K.K. Dewan, On the zeros of a polynomial, Numerical Methods and Approximation Theory 111, $Ni\acute{s}$(1987), 121-128.

[4] M.Dehmer, On the location of zeros of complex polynomials, Jipam, 4(2006), 1-13.

[5] N.K.Govil and Q. I. Rahman, On the Eneström – Kakeya theorem, Tohoku Math. J., 20(1968), 126-136

[6] E. Kittaneh, Bounds for the zeros of polynomials from matrix inequalities, Arch. Math., 81(2003), 601-608

[7] M. Marden , Geometry of polynomials, IInd ed. Mathematical Surveys , No 3 , Amer . math. Soc., Providence R.I., 1966.

[8] G.V. Milovanovic, D. S. Mitrinovic and Th. M Rassias, Topics in polynomials, Extremal problems, Inequalities, Zeros, World Scientific Publishing Co., Singapore, 1994.

[9] Q. G. Mohammad,, On the zeros of polynomials, Amer . Math. Monthly, 72 (1965) 631-633

[10] Q.I .Rahman and G. Schmeisser, Analytic theory of Polynomials, oxford University Press, New, York,(2002)

[11] W.M.Shah and A Liman, On Enestrom –Kakeya theorem and related analytic functions, Proc. Indian Acad. Sci(Math. Sci), 117(3) (2007), 359-370.

[12] Ghulshan Singh and W.M.Shah, On the Enestrom –Kakeya theorem, Applied Mathematics, 3(2010), 555-560

[13] Ghulshan Singh and W.M.Shah, Number of Zeros of a Polynomial in a given Region, International Journal of Modern Mathematical Sciences, 2012, 1(2):103-109

[14] M.A.Kawoosa and W.M.Shah, Bounds for the Zeros of a Polynomial with Perturbed Coefficients, International Journal of Modern Mathematical Sciences, 2012, 3(1): 55-62.

Source of Support: Nil, Conflict of interest: None Declared