

PARTIALLY ORDERED FILTERS IN PARTIALLY ORDERED SEMIGROUPS

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ABSTRACT

In this paper the terms, *po filter*, *po filter of a po semigroup generated by a subset* and *principle po filter generated by an element a in a po semigroup* are introduced. It is proved that in a po semigroup S , the nonempty intersection of a family of po filters is also a po filter of S . It is also proved that a nonempty subset F of a po semigroup S is a po filter if and only if S/F is a completely prime po ideal of S or empty. It is proved that every po filter F of a po semigroup S is a i) *po-c-system* of S ii) *po-m-system* of S and iii) *po-d-system* of S . It is proved that the po filter of a po semigroup S generated by a nonempty subset A is the intersection of all po filters of S containing A . Let S be a po semigroup and $a \in S$, then it is proved that $N(a)$ is the least filter of S containing $\{a\}$.

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Key Words: *po filter*, *po filter of a po semigroup generated by a subset* and *principle po filter generated by an element a in a po semigroup*.

1. INTRODUCTION

The algebraic theory of semigroups was widely studied by CLIFFORD [2], [3], PETRICH [4] and LJAPIN [5]. The ideal theory in general semigroups was developed by ANJANEYULU [1]. In this paper we introduce the notions of po filter, po filter of a po semigroup generated by a subset and principle po filter generated by an element a in a po semigroup and characterize po filters, po filters of a po semigroup generated by a subset and principle po filters generated by an element a in a po semigroup. in po semigroups.

2. PRELIMINARIES

Definition 2.1: A semigroup S is said to be a *partially ordered semigroup* if S is a partially ordered set such that $a \leq b \Rightarrow ax \leq bx$, $xa \leq xb$ for all $a, b, x \in S$.

Definition 2.2: A po (left/right) ideal A of a po semigroup S is said to be a *completely prime (left/right) ideal* of S provided $x, y \in S$ and $xy \in A$ implies either $x \in A$ or $y \in A$.

Definition 2.3: A po ideal A of a po semigroup S is said to be a *prime ideal* of S provided X, Y are ideals of S and $XY \subseteq A \Rightarrow X \subseteq A$ or $Y \subseteq A$.

Definition 2.4: A nonempty subset A of a po semigroup S is said to be an *m-system* provided for any $a, b \in A$ implies that $(S^1 a S^1 b S^1) \cap A \neq \emptyset$.

Definition 2.5: A po ideal A of a po semigroup S is said to be a *completely semiprime po ideal* provided $x \in S$, $x^n \in A$ for some natural number n implies $x \in A$.

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Definition 2.6: Let S be a po semigroup. A nonempty subset A of S is said to be a **d -system** of S if $a \in A \Rightarrow a^n \in A$ for all natural number n .

Definition 2.7: A po ideal A of a po semigroup S is said to be **semiprime po ideal** provided X is po ideal of S and $X^n \subseteq A$ for some natural number n implies $X \subseteq A$.

Definition 2.8: A non-empty subset A of a po semigroup S is said to be an **n -system** provided $a \in A$ implies that $(S^1 a S^1) \cap A \neq \emptyset$.

Theorem 2.9: Every completely prime po ideal of a po semigroup S is a prime po ideal of S .

Theorem 2.10: Let S be a commutative po semigroup. A po ideal A of S is a prime po ideal if and only if A is a completely prime po ideal.

Theorem 2.11: Every m -system in a po semigroup S is an n -system.

Theorem 2.12: A po ideal Q of a po semigroup S is a semiprime po ideal if and only if $S \setminus Q$ is an n -system of S (or) empty.

Theorem 2.13: Every completely prime po ideal of a po semigroup S is a completely semiprime po ideal of S .

Theorem 2.14: A po ideal A of a po semigroup S is completely semiprime if and only if $S \setminus A$ is a d -system of S or empty.

Theorem 2.15: Let S be a commutative po semigroup. A po ideal A of S is completely semiprime if and only if it is semiprime.

Theorem 2.16: If N is an n -system in a po semigroup S and $a \in N$, then there exist an m -system M in S such that $a \in M$ and $M \subseteq N$.

3. PO FILTERS IN PO SEMIGROUPS

Definition 3.1: A po sub semigroup F of a po semigroup S is said to be a **po left filter** of S if

- (1) $a, b \in S, ab \in F \Rightarrow a \in F$.
- (2) $a, b \in T, a \leq b$ and $a \in F \Rightarrow b \in F$.

Note 3.2: A po subsemigroup F of a po semigroup S is a **po left filter** of S iff

- (1) $a, b \in S, ab \in F \Rightarrow a \in F$.
- (2) $(F) \subseteq F$.

Theorem 3.3: The nonempty intersection of two po left filters of a po semigroup S is also a po left filter of S .

Proof: Let A, B be two po left filters of S .

Let $a, b \in S, ab \in A \cap B$.

$ab \in A \cap B \Rightarrow ab \in A$ and $ab \in B$.

$a, b \in S, ab \in A, A$ is a po left filter of $S \Rightarrow a \in A$.

$a, b \in S, ab \in B, B$ is a po left filter of $S \Rightarrow a \in B$.

$a \in A, a \in B \Rightarrow a \in A \cap B$.

$a, b \in S, ab \in A \cap B \Rightarrow a \in A \cap B$.

Let $a, b \in S, a \leq b$ and $a \in A \cap B$. Now $a \in A \cap B \Rightarrow a \in A, a \in B$.

$a, b \in S, a \leq b, a \in A, A$ is a po left filter of $S \Rightarrow b \in A$.

$a, b \in S, a \leq b, a \in B, B$ is a po left filter of $S \Rightarrow b \in B$.

$b \in A, b \in B \Rightarrow b \in A \cap B$. Thus $a, b \in S, a \leq b, a \in A \cap B \Rightarrow b \in A \cap B$.

Therefore $A \cap B$ is a po left filter of S .

Theorem 3.4: The nonempty intersection of a family of po left filters of a po semigroup S is also a po left filter of S .

Proof: Let $\{F_\alpha\}_{\alpha \in \Delta}$ be a family of po left filters of S and let $F = \bigcap_{\alpha \in \Delta} F_\alpha$.

Let $a, b \in S, ab \in F$. Now $ab \in F \Rightarrow ab \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow ab \in F_\alpha$ for each $\alpha \in \Delta$.

$ab \in F_\alpha, F_\alpha$ is a po left filter of $S \Rightarrow a \in F_\alpha$ for each $\alpha \in \Delta \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow a \in F$.

Let $a, b \in S, a \leq b$ and $a \in F$. Now $a \in F \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow a \in F_\alpha$ for each $\alpha \in \Delta$

$a, b \in S, a \leq b, a \in F_\alpha, F_\alpha$ is a po left filter of $S \Rightarrow b \in F_\alpha$ for all $\alpha \in \Delta$.

$\Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F$. Therefore F is a po left filter of S .

Theorem 3.5: A nonempty subset F of a po semigroup S is a po left filter if and only if $S \setminus F$ is a completely prime po right ideal of S or empty.

Proof: Suppose that F is a po left filter of S and $S \setminus F \neq \emptyset$. Let $b \in S$ and $a \in S \setminus F$.

Now $a \in S \setminus F \Rightarrow a \notin F$.

If $ab \in F$, then since F is a po left filter of $S, a \in F$. It is a contradiction.

Therefore $ab \notin F$. Hence $ab \in S \setminus F$.

Let $a \in S \setminus F$ and $s \in S$ such that $s \leq a$.

If $s \in F$ then $s \leq a, F$ is a po left filter of $S \Rightarrow a \in F$. It is a contradiction.

Therefore $s \in S \setminus F$. Thus $S \setminus F$ is a po right ideal of S .

Let $a, b \in S$ and $ab \in S \setminus F$. Now $ab \notin F$.

Suppose if possible $a \notin S \setminus F, b \notin S \setminus F$.

Then $a \in F, b \in F \Rightarrow ab \in F$. It is a contradiction.

Therefore either $a \in S \setminus F$ or $b \in S \setminus F$.

Hence $S \setminus F$ is completely prime.

Therefore $S \setminus F$ is a completely prime po right ideal of S .

Conversely suppose that $S \setminus F$ is a completely prime po right ideal of S or empty.

If $S \setminus F = \emptyset$, then $F = S$. Thus F is a po left filter of S .

Assume that $S \setminus F \neq \emptyset$. Let $a, b \in S$, and $ab \in F$. If $a \notin F$ then $a \in S \setminus F$.

$b \in S, a \in S \setminus F, S \setminus F$ is a po right ideal of $S \Rightarrow ab \in S \setminus F \Rightarrow ab \notin F$. It is a contradiction. Thus $a \in F$.

Let $a, b \in S, a \leq b$ and $a \in F$. If $b \notin F$, then $b \in S \setminus F$.

$a, b \in S, b \in S \setminus F, a \leq b, S \setminus F$ is a po right ideal of $S \Rightarrow a \notin F$. It is a contradiction.

Thus $b \in F$. Therefore F is a po left filter of S .

Corollary 3.6: Let S be a po semigroup and F is a po left filter of S . Then $S \setminus F$ is a prime po right ideal of S or empty.

Proof: Since F is a po left filter, by theorem 3.5, $S \setminus F$ is a completely prime po right ideal of S or empty. By theorem 2.9, $S \setminus F$ is a prime po right ideal of S or empty.

Definition 3.7: A subsemigroup F of a po semigroup S is said to be *po right filter* of S if

- (1) $a, b \in S, ab \in F \Rightarrow b \in F$
- (2) $a, b \in S, a \leq b$ and $a \in F \Rightarrow b \in F$.

Note 3.8: A po subsemigroup F of a po semigroup S is a *po right filter* of S if

- (1) $a, b \in S, ab \in F \Rightarrow b \in F$.
- (2) $(F) \subseteq F$.

Theorem 3.9: The nonempty intersection of two po right filters of a po semigroup S is also a po right filter.

Proof: Let A, B be two po right filters of S .

Let $a, b \in S, ab \in A \cap B$.

$ab \in A \cap B \Rightarrow ab \in A$ and $ab \in B$.

$a, b \in S, ab \in A, A$ is a po right filter of $S \Rightarrow b \in A$.

$a, b \in S, ab \in B, B$ is a po right filter of $S \Rightarrow b \in B$.

$b \in A, b \in B \Rightarrow b \in A \cap B$.

$a, b \in S, ab \in A \cap B \Rightarrow b \in A \cap B$.

Let $a, b \in S, a \leq b$ and $a \in A \cap B$. Now $a \in A \cap B \Rightarrow a \in A, a \in B$.

$a, b \in S, a \leq b, a \in A, A$ is a po right filter of $S \Rightarrow b \in A$.

$a, b \in S, a \leq b, a \in B, B$ is a po right filter of $S \Rightarrow b \in B$.

$b \in A, b \in B \Rightarrow b \in A \cap B$. Thus $a, b \in S, a \leq b, a \in A \cap B \Rightarrow b \in A \cap B$.

Therefore $A \cap B$ is a po right filter of S .

Theorem 3.10: The nonempty intersection of a family of po right filters of a po semigroup S is also a po right filter.

Proof: Let $\{F_\alpha\}_{\alpha \in \Delta}$ be a family of po right filters of S and let $F = \bigcap_{\alpha \in \Delta} F_\alpha$.

Let $a, b \in S, ab \in F$. Now $ab \in F \Rightarrow ab \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow ab \in F_\alpha$ for each $\alpha \in \Delta$.

$ab \in F_\alpha, F_\alpha$ is a po right filter of $S \Rightarrow b \in F_\alpha$.

Let $a, b \in S, a \leq b$ and $a \in F$. Now $a \in F \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow a \in F_\alpha$ for each $\alpha \in \Delta$

$a, b \in S, a \leq b, a \in F_\alpha, F_\alpha$ is a po right filter of $S \Rightarrow b \in F_\alpha$ for all $\alpha \in \Delta$.

$\Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F$.

Therefore F is a po right filter of S .

Theorem 3.11: A nonempty subset F of a po semigroup S is a po right filter if and only if $S \setminus F$ is a completely prime po left ideal of S or empty.

Proof: Suppose that F is a po right filter of S and $S \setminus F \neq \emptyset$. Let $b \in S$ and $a \in S \setminus F$.

Now $a \in S \setminus F \Rightarrow a \notin F$.

If $ba \in F$, then since F is a po right filter of S , $a \in F$. It is a contradiction.

Therefore $ba \notin F$. Hence $ba \in S \setminus F$.

Let $a \in S \setminus F$ and $s \in S$ such that $s \leq a$.

If $s \in F$ then $s \leq a$, F is a po right filter of $S \Rightarrow a \in F$. It is a contradiction.

Therefore $s \in S \setminus F$. Thus $S \setminus F$ is a po left ideal of S .

Let $a, b \in S$ and $ab \in S \setminus F$. Now $ab \notin F$.

Suppose if possible $a \notin S \setminus F$, $b \notin S \setminus F$.

Then $a \in F$, $b \in F \Rightarrow ab \in F$. It is a contradiction.

Therefore either $a \in S \setminus F$ or $b \in S \setminus F$.

Hence $S \setminus F$ is completely prime.

Therefore $S \setminus F$ is a completely prime po left ideal of S .

Conversely suppose that $S \setminus F$ is a completely prime po left ideal of S or empty.

If $S \setminus F = \emptyset$, then $F = S$. Thus F is a po right filter of S .

Assume that $S \setminus F \neq \emptyset$. Let $a, b \in S$ and $ab \in F$. If $b \notin F$ then $b \in S \setminus F$.

$a \in S$, $b \in S \setminus F$, $S \setminus F$ is a po left ideal of $S \Rightarrow ab \in S \setminus F \Rightarrow ab \notin F$. It is a contradiction.

Thus $b \in F$.

Let $a, b \in S$, $a \leq b$ and $a \in F$. If $b \notin F$, then $b \in S \setminus F$.

$a, b \in S$, $b \in S \setminus F$, $a \leq b$, $S \setminus F$ is a po left ideal of $S \Rightarrow a \in S \setminus F$

$\Rightarrow a \notin F$. It is a contradiction.

Therefore F is a po right filter of S .

Corollary 3.12: Let S be a po semigroup and F is a po right filter. Then $S \setminus F$ is a prime po left ideal of S or empty.

Proof: Since F is a po right filter. By theorem 3.11, $S \setminus F$ is a completely prime left po ideal of S or empty. By theorem 2.9, $S \setminus F$ is a prime po left ideal of S or empty.

Definition 3.13: A po subsemigroup F of a po semigroup S is said to be *po filter* of S if

- (1) $a, b \in S$, $ab \in F \Rightarrow a, b \in F$
- (2) $a, b \in S$, $a \leq b$ and $a \in F \Rightarrow b \in F$

Note 3.14: A po subsemigroup F of a po semigroup S is a *po filter* of S iff

- (1) $a, b \in S$, $ab \in F \Rightarrow a, b \in F$
- (2) $(F) \subseteq F$.

Note 3.15: A po sub semi group F of a po semigroup S is a *po filter* of S iff F is a po left filter, a po right filter and po lateral filter of S .

Definition 3.16: A po filter F of a po semigroup S is said to be a *proper po filter* if $F \neq S$.

Theorem 3.17: The nonempty intersection of two po filters of a po semigroup S is also a po filter of S .

Proof: Let A, B be two po filters of S .

Let $a, b \in S, ab \in A \cap B$.

$ab \in A \cap B \Rightarrow ab \in A$ and $ab \in B$.

$a, b \in S, ab \in A, A$ is a po filter of $S \Rightarrow a, b \in A$.

$a, b \in S, ab \in B, B$ is a po filter of $S, \Rightarrow a, b \in B$.

$a, b \in A, a, b \in B \Rightarrow a, b \in A \cap B$.

Let $a, b \in S, a \leq b$ and $a \in A \cap B$. Now $a \in A \cap B \Rightarrow a \in A, a \in B$.

$a, b \in S, a \leq b, a \in A, A$ is a po filter of $S \Rightarrow b \in A$.

$a, b \in S, a \leq b, a \in B, B$ is a po filter of $S \Rightarrow b \in B$.

$b \in A, b \in B \Rightarrow b \in A \cap B$. Thus $a, b \in S, a \leq b$ and $a \in A \cap B \Rightarrow b \in A \cap B$.

Therefore $A \cap B$ is a po filter of S .

Theorem 3.18: The nonempty intersection of a family of po filters of a po semigroup S is also a po filter of S .

Proof: Let $\{F_\alpha\}_{\alpha \in \Delta}$ be a family of po filters of S and let $F = \bigcap_{\alpha \in \Delta} F_\alpha$.

Let $a, b \in S, ab \in F$. Now $ab \in F \Rightarrow ab \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow ab \in F_\alpha$ for each $\alpha \in \Delta$.

$ab \in F_\alpha, F_\alpha$ is a po filter of $S \Rightarrow a, b \in F_\alpha$.

Let $a, b \in S, a \leq b$ and $a \in F$. Now $a \in F \Rightarrow a \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow a \in F_\alpha$ for each $\alpha \in \Delta$

$a, b \in S, a \leq b$ and $a \in F_\alpha, F_\alpha$ is a po filter of $S \Rightarrow b \in F_\alpha$ for all $\alpha \in \Delta$.

$\Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F$.

Therefore F is a po filter of S .

Theorem 3.19: A nonempty subset F of a po semigroup S is a po filter if and only if $S \setminus F$ is a completely prime po ideal of S or empty.

Proof: Suppose that F is a po filter of S and $S \setminus F \neq \emptyset$. Let $b \in S$ and $a \in S \setminus F$.

Now $a \in S \setminus F \Rightarrow a \notin F$.

If $ab \in F$, then since F is a po filter of $S, a \in F$. It is a contradiction.

Therefore $ab \notin F$. Hence $ab \in S \setminus F$.

Similarly $ba \in S \setminus F$.

Let $a \in S \setminus F$ and $s \in S$ such that $s \leq a$.

If $s \in F$ then $s \leq a, F$ is a po filter of $S \Rightarrow a \in F$. It is a contradiction.

Therefore $s \in S \setminus F$. Thus $S \setminus F$ is a po ideal of S .

Let $a, b \in S$ and $ab \in S \setminus F$. Now $ab \notin F$.

Suppose if possible $a \notin S \setminus F, b \notin S \setminus F$.

Then $a \in F, b \in F \Rightarrow ab \in F$. It is a contradiction.

Therefore either $a \in S \setminus F$ or $b \in S \setminus F$.

Hence $S \setminus F$ is completely prime.

Therefore $S \setminus F$ is a completely prime po ideal of S .

Conversely suppose that $S \setminus F$ is a completely prime po ideal of S or empty.

If $S \setminus F = \emptyset$, then $F = S$. Thus F is a po filter of S .

Assume that $S \setminus F \neq \emptyset$. Let $a, b \in S$ and $ab \in F$. If $a \notin F$ then $a \in S \setminus F, b \in S, a \in S \setminus F, S \setminus F$ is a po ideal of $S \Rightarrow ab \in S \setminus F \Rightarrow ab \notin F$. It is a contradiction.

Thus $a \in F$.

Similarly $b \in F$.

Let $a, b \in S, a \leq b$ and $a \in F$. If $b \notin F$, then $b \in S \setminus F$.

$a, b \in S, b \in S \setminus F, a \leq b, S \setminus F$ is a po ideal of $S \Rightarrow a \in S \setminus F \Rightarrow a \notin F$. It is a contradiction.

Therefore F is a po filter of S .

Corollary 3.20: Let S be a po semigroup. If F is a po filter, then $S \setminus F$ is a prime po ideal of S or empty.

Proof: Since F is a po filter of S . By theorem 3.19, $S \setminus F$ is a completely prime po ideal of S or empty. By theorem 2.9, $S \setminus F$ is a prime po ideal of S or empty.

Corollary 3.21: A nonempty subset F of a commutative po semigroup S is a po filter if and only if $S \setminus F$ is a prime po ideal of S or empty.

Proof: Suppose that $S \setminus F$ is po filter of commutative po semigroup S .

By corollary 3.20, $S \setminus F$ is prime po ideal of S or empty.

Conversely suppose that $S \setminus F$ is a prime po ideal of S or empty.

If $S \setminus F = \emptyset$, then $F = S$. Thus F is a po filter of S .

Assume that $S \setminus F$ is a prime po ideal of S .

By theorem 2.10, $S \setminus F$ is a completely prime po ideal of S or empty.

By theorem 3.19, F is a po filter of S .

Theorem 3.22: Every po filter F of a po semigroup S is a po- c -system of S .

Proof: Suppose that F is a po filter.

By theorem 3.19, $S \setminus F$ is a completely prime po ideal of S .

By theorem 2.12, F is a po- c -system of S .

Theorem 3.23: A po semigroup S does not contain proper po filters if and only if S does not contain proper completely prime po ideals.

Proof: Suppose that a po semigroup S does not contain proper po filters.

Let A be a completely prime po ideal of S and $A \subset S$.

Then $\emptyset \neq S \setminus A \subseteq S$ and $S \setminus (S \setminus A) (= A)$ is a completely prime po ideal of S .

Since $S \setminus A$ is the complement of A to S , by theorem 3.19, $S \setminus A$ is a po filter of S .

Then $S \setminus A = S$ and hence $A = \emptyset$. It is a contradiction.

Therefore S does not contain proper completely prime po ideals.

Conversely suppose that S does not contain proper completely prime po ideals.

Let F be a po filter of S and $F \subset S$.

Since $S \setminus F \neq \emptyset$, by theorem 3.19, $S \setminus F$ is a completely prime po ideal of S .

Then $S \setminus F = S$ and hence $F = \emptyset$. It is a contradiction.

Therefore S does not contain proper po filters.

Theorem 3.24: Every po filter F of a po semigroup S is a po- m -system of S .

Proof: Suppose that F is a po filter of a po semigroup S .

By corollary 3.20, $S \setminus F$ is a prime po ideal of S .

By theorem 2.12, $S \setminus (S \setminus F) = F$ is a po- m -system of S or empty.

Corollary 3.25: Let S be a po semigroup. If F is a po filter, then $S \setminus F$ is a completely semiprime po ideal of S .

Proof: Suppose that F is a po filter of a po semigroup S .

By theorem 3.20, $S \setminus F$ is a completely prime po ideal of S .

By theorem 2.13, $S \setminus F$ is a completely semiprime po ideal of S .

Corollary 3.26: Every po filter F of a po semigroup S is a po- d -system of S .

Proof: Suppose that F is a po filter of a po semigroup S .

By corollary 3.25, $S \setminus F$ is a completely semiprime po ideal of S .

By theorem 2.14, $S \setminus (S \setminus F) = F$ is a po- d -system of S or empty.

Corollary 3.27: Let S be a po semigroup. If F is a po filter, then $S \setminus F$ is a semiprime po ideal of S .

Proof: Suppose that F is a po filter of a po semigroup S .

By theorem 3.19, $S \setminus F$ is a completely prime po ideal of S .

By theorem 2.13, $S \setminus F$ is a completely semiprime po ideal of S .

By theorem 2.15, $S \setminus F$ is a semiprime po ideal of S .

Corollary 3.28: Every po filter F of a po semigroup S is a po- n -system of S .

Proof: Suppose that F is a po filter of a po semigroup S .

By corollary 3.27, $S \setminus F$ is a semiprime po ideal of S .

By theorem 2.16, $S \setminus (S \setminus F) = F$ is a po- n -system of S .

Definition 3.29: Let S be a po semigroup and A be a nonempty subset of S . The smallest po left filter of S containing A is called *po left filter of S generated by A* and it is denoted by $F_l(A)$.

Theorem 3.30: The po left filter of a po semigroup S generated by a nonempty subset A of S is the intersection of all po left filters of S containing A .

Proof: Let Δ be the set of all po left filters of S containing A .

Since S itself is a po left filter of S containing A , $S \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 3.4, F^* is a po left filter of S .

Let K be a po left filter of S containing A .

Clearly $A \subseteq K$ and K is a po left filter of S .

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$. Therefore F^* is the smallest po left filter of S containing A and hence F^* is the po left filter of S generated by A .

Definition 3.31: Let S be a po semigroup and A be a nonempty subset of S . The smallest po right filter of S containing A is called *po right ideal of S generated by A* and it is denoted by $F_r(A)$.

Theorem 3.32: The po right filter of a po semigroup S generated by a nonempty subset A is the intersection of all po right filters of S containing A .

Proof: Let Δ be the set of all po right filters of S containing A .

Since S itself is a po right filter of S containing A , $S \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 3.11, F^* is a po right filter of S .

Let K be a po right filter of S containing A .

Clearly $A \subseteq K$ and K is a po right filter of S .

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$. Therefore F^* is the smallest po right filter of S containing A and hence F^* is the po right filter of S generated by A .

Definition 3.33: Let S be a po semigroup and A be a nonempty subset of S . The smallest po filter of S containing A is called *po filter of S generated by A* and it is denoted by $N(A)$.

Theorem 3.34: The po filter of a po semigroup S generated by a nonempty subset A is the intersection of all po filters of S containing A .

Proof: Let Δ be the set of all po filters of S containing A .

Since S itself is a po filter of S containing A , $S \in \Delta$. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{F \in \Delta} F$. Since $A \subseteq F$ for all $F \in \Delta$, $A \subseteq F^*$. So $F^* \neq \emptyset$.

By theorem 3.15, F^* is a po filter of S.

Let K be a po filter of S containing A.

Clearly $A \subseteq K$ and K is a po filter of S.

Therefore $K \in \Delta \Rightarrow F^* \subseteq K$.

Therefore F^* is the po filter of S generated by A.

Definition 3.35: A po filter F of a po semigroup S is said to be a *principal po filter* provided F is a po filter generated by $\{a\}$ for some $a \in S$. It is denoted by $N(a)$.

Corollary 3.36: Let S be a po semigroup and $a \in S$. Then $N(a)$ is the least filter of S containing $\{a\}$.

Note 3.37: For every $a \in S$, the intersection of all po filters containing $\{a\}$ is again a po filter and thus the least po filter containing $\{a\}$.

Theorem 3.38: If $N(b) \subseteq N(a)$, then $N(a) \setminus N(b)$, if it is nonempty, is a completely prime po ideal of $N(a)$.

Proof: Clearly $N(b)$ is a po filter of $N(a)$, By theorem 3.19, $N(a) \setminus N(b)$ is a completely prime po ideal of $N(a)$.

Theorem 3.39: If $a, b \in S$ and $b \in N(a)$, then $N(b) \subseteq N(a)$.

Proof: From the definition of the principal po filter, it is clear.

Corollary 3.40: If $a, b \in S$ and $a \leq b$ then $N(b) \subseteq N(a)$.

Proof: Since $a \leq b$ then it is clear that $b \in N(a)$.

By theorem 3.39, we have $N(b) \subseteq N(a)$.

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