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GENERALISATIONS OF STRONG AND ABSOLUTE ALMOST CONVERGENCE WITH AN INDEX $k \geq 1$

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ABSTRACT

The object is to prove generalisations of strong and absolute almost convergence with an Index ≥ 1.

Keywords: *Strong and absolute convergence, absolute almost convergence, Banach limits, Sublinear functionals, paranorm etc.*

1. INTRODUCTION AND NOTATIONS

Let l_∞ be the set of all bounded real sequences $x = (x_n)$ with norm $\|x\| = sup|x_n|$. Given an infinite series $\sum a_n$, denoted by 'a'.Let $x_n = a_0 + a_1 + \cdots + a_n$.

Lorentz [9] defined a sequence $x \in l_\infty$ to be almost convergent to a number s if all its Banach limits coincide at s and also proved that "a sequence x is almost convergent to s if and only if

$$t_{kn} = t_{kn}(x) = \frac{1}{k+1}\sum_{i=0}^{k} x_{n+i} \to s \tag{1.1}$$

as $k \to \infty$ uniformly in n"

Maddox [11] has defined $x \in l_\infty$ to be strongly almost convergent to a number s if

$$t_{kn}(|x-s|) = \frac{1}{k+1}\sum_{i=0}^{k}|x_{n+i} - s| \to 0 \text{ as } k \to \infty \text{ uniformly in } n \tag{1.2}$$

f denotes the set of all almost convergent sequences and $[f]$ denotes the set of all strongly almost convergent sequences.

A sequence x is said to be absolutely almost convergent if

$$\sum_{k=0}^{\infty}|t_{kn} - t_{k-1,n}| \text{ converges uniformly in } n \tag{1.3}$$

$\hat{l}$ denotes the set of all absolutely almost convergent sequences .

We write

$$d_{mn} = d_{mn}(x) = \frac{1}{m+1}\sum_{k=0}^{m} t_{kn}(x) \tag{1.4}$$

Also g_{mn} has been defined [10] as

$$g_{mn} = g_{mn}(x) = \frac{1}{m+1}\sum_{k=0}^{m} d_{kn}(x) \tag{1.5}$$

where $g_{-1,n} = d_{-1,n} = t_{-1,n} = x_{n-1}$

The following sequence spaces have been introduced [10] and their relative strengths have been studied in details.

Let

$$u = \{x: \frac{1}{m+1}\sum_{k=0}^{m} d_{kn}(x-s) \to 0 \text{ as } m \to \infty \text{ uniformly in } n, \text{for some } s\} \tag{1.6}$$

$$[u] = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m}|d_{kn}(x-s)| \to 0 \text{ as } m \to \infty \text{ uniformly in } n, \text{for some } s\right\} \tag{1.7}$$

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$$[u_1] = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m} d_{kn}\,(|x-s|) \to 0 \; as \; m \to \infty \; uniformly \; in \; n, for \; some \; s\right\} \quad (1.8)$$

$$v = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m} g_{kn}\,(x-s) \to 0 \; as \; m \to \infty \; uniformly \; in \; n, for \; some \; s\right\} \quad (1.9)$$

$$[v] = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m} |g_{kn}\,(x-s)| \to 0 \; as \; m \to \infty \; uniformly \; in \; n, for \; some \; s\right\} \quad (1.10)$$

$$[v_1] = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m} g_{kn}\,(|x-s|) \to 0 \; as \; m \to \infty \; uniformly \; in \; n, for \; some \; s\right\} \quad (1.11)$$

$$\hat{u} = \left\{x: \sum_{k=0}^{\infty} \left|g_{kn} - g_{k-1,n}\right| \; convergent \; uniformly \; in \; n\right\} \quad (1.12)$$

$$\hat{\hat{u}} = \left\{x: \sup_{n} \sum_{k=0}^{m} \left|g_{kn} - g_{k-1,n}\right| < \infty\right\} \quad (1.13)$$

$$D_2 = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m} d_{k0}(x) \to s \; as \; m \to s\right\} \quad (1.14)$$

$$[D_2] = \left\{x: \frac{1}{m+1}\sum_{k=0}^{m} |d_{k0}(x) - s| \to 0 \; as \; m \to \infty\right\} \quad (1.15)$$

Hence it may be remarked that the spaces D_2 and $[D_2]$ can be called Cesaro Summable sequences of order 2 and strongly Cesaro Summable sequence of order 2 respectively.

It has been proved [10] that $\hat{u} \subseteq \hat{\hat{u}}$.

2. SEQUENCE SPACES WITH INDEX

The main object of this chapter is to provide a suitable extension of sequence spaces $\hat{u}$ *and* $\hat{\hat{u}}$ to the respective spaces with index $k \geq 1$.

These spaces can be generalized by giving an index to some of them as follows.

Let

$$P_k = \left\{x : \sum_{m=1}^{\infty} m^{k-1}\left|g_{mn} - g_{m-1,n}\right|^k \; converges \; uniformly \; in \; n\right\} \quad (2.1)$$

and

$$Q_k = \left\{x : \sup_{n} \sum_{m=1}^{\infty} m^{k-1}\left|g_{mn} - g_{m-1,n}\right|^k < \infty\right\} \text{ where } k \geq 1 \quad (2.2)$$

The following spaces have been defined and their relative strengths have also been studied there.

$$A_k = \left\{x : \sum_{m=1}^{\infty} m^{k-1}\left|d_{mn} - d_{m-1,n}\right|^k \; converges \; uniformly \; in \; n\right\} \quad (2.3)$$

and

$$B_k = \left\{x : \sup_{n} \sum_{m=1}^{\infty} m^{k-1}\left|d_{mn} - d_{m-1,n}\right|^k < \infty\right\} \quad (2.4)$$

RESULTS

Now we proceed to examine the relative strengths of these spaces.

Theorem 1: $G_k \subseteq A_k \subseteq P_k \subseteq Q_k$

Proof: The inclusion $G_k \subseteq A_k$ has been proved. So we proceed to prove

(i) $A_k \subseteq P_k$ and
(ii) $P_k \subseteq Q_k$

Let $x \in A_{k,}$ then by definition

$$\sum_{m=1}^{\infty} m^{k-1}\left|d_{mn} - d_{m-1,n}\right|^k \; converges \; uniformly \; in \; n$$

We consider

$$\sum_{m=1}^{\infty} m^{k-1}\left|g_{mn} - g_{m-1,n}\right|^k$$

Since

$$g_{mn} - g_{m-1,n} = \frac{1}{m(m+1)}\sum_{v=1}^{\infty} v(\,d_n - d_{-1,n})$$

We have by Holders inequality

$$\left|g_{mn} - g_{m-1,n}\right|^k \leq \frac{1}{m(m+1)^k} \sum_{v=1}^{m} v^k \left|d_{vn} - d_{v-1,n}\right|^k$$

So

$$\sum_{m=1}^{\infty} m^{k-1}\left|g_{mn} - g_{m-1,n}\right|^k \leq \sum_{m=1}^{\infty} m^{k-1}\left(\frac{1}{m(m+1)}\sum_{v=1}^{m} v^k \left|d_{vn} - d_{v-1,n}\right|^k\right)$$
$$\leq \sum_{v=1}^{\infty} v^k \left|d_{vn} - d_{v-1,n}\right|^k \sum_{m=v}^{\infty} \frac{1}{m(m+1)}$$
$$\leq \sum_{v=1}^{\infty} v^{k-1} \left|d_{vn} - d_{v-1,n}\right|^k$$

But then $\sum_{v=1}^{\infty} v^{k-1} \left|d_{vn} - d_{v-1,n}\right|^k$ converges uniformly in n, implies that $\sum_{v=1}^{\infty} m^{k-1} \left|g_{mn} - g_{m-1,n}\right|^k$ converges uniformly in n.

Hence

$$A_k \subseteq P_k \tag{2.5}$$

(ii) $P_k \subseteq Q_k$

Let $x \in P_k$. Then by definition

$\sum_{m=1}^{\infty} m^{k-1} \left|g_{mn} - g_{m-1,n}\right|^k$ converges uniformly in n

So there exits $M > 0$ such that

$$\sum_{m \geq M} m^{k-1} \left|g_{mn} - g_{m-1,n}\right|^k < 1 \text{ for all } n \tag{2.6}$$

To prove $P_k \subseteq Q_k$ it is enough to show that

$\sum_{m=1}^{M-1} m^{k-1} \left|g_{mn} - g_{m-1,n}\right|^k = 0(1)$ for all n

From (2.6) it follows that for $m \geq M$

$\left|g_{mn} - g_{m-1,n}\right| \leq \frac{1}{m^{k-1}} < 1$ for all n

This implies that for fixed m

$$\left|g_{mn} - g_{m-1,n}\right| < 1 \text{ for all } n \tag{2.7}$$

From the definition of d_{mn} we can see that

$$d_{mn}(x) = (m+1)g_{mn}(x) - m\, g_{m-1,n}(x)$$

Therefore,

$$d_{mn}(x) - d_{m-1,n}(x) = \{(m+1)g_{mn}(x) - mg_{m-1,n}(x)\} - \{m\, g_{m-1,n}(x) - (m-1)g_{m-2,n}(x)\}$$
$$= mg_{mn}(x) + g_{mn}(x) - 2m\, g_{m-1,n}(x) + (m-1)\, g_{m-2,n}(x)$$
$$= mg_{mn}(x) - mg_{m-1,n}(x) + g_{mn}(x) - g_{m-1,n}(x) - (m-1)\, g_{m-1,n}(x) + (m-1)\, g_{m-2,n}(x)$$
$$= m\{g_{mn}(x) - g_{m-1,n}(x)\} + \{g_{mn}(x) - g_{m-1,n}(x)\} - (m-1)\{g_{m-1,n}(x) - g_{m-2,n}(x)\}$$

From (2.7) we have for fixed $m \geq M$ that

$\left|d_{mn}(x) - d_{m-1,n}(x)\right| < 1$ for all n

But since

$$\left|d_{mn}(x) - d_{m-1,n}(x)\right| = \frac{1}{m(m+1)}\sum_{k=0}^{m} k(t_{tn} - t_{k-1,n})$$

We have

$$(m+1)\left(d_{mn} - d_{m-1,n}\right) - (m-1)\left(d_{m-1,n} - d_{m-2,n}\right) = t_{mn} - t_{m-1,n}$$

From which it follows that

$\left|t_{mn} - t_{m-1,n}\right| < 1$ for all $n, m \geq M$

Again since

$$t_{mn} - t_{m-1,n} = \frac{1}{m(m+1)}\sum_{v=1}^{m} v\; a_{v+n}$$

and

$$(m+1)\left(t_{mn} - t_{m-1,n}\right) - (m-1)\left(t_{m-1,n} - t_{m-2,n}\right) = a_{n+m}$$

So for fixed $m \geq M, |a_{n+m}| = 0(1)$ for all n and (a_k) is bounded.

Thus

$$\begin{aligned}|t_{mn} - t_{m-1,n}| = |\phi_{mn}| &\leq \frac{1}{m(m+1)} \sum v\, |a_{v+n}| \\ &= 0(1) \frac{1}{m(m+1)} \sum v \\ &= 0(1) \quad \text{for all } n, m \geq M\end{aligned}$$

Hence

$$\begin{aligned}|d_{mn} - d_{m-1,n}| &= \frac{1}{m(m+1)} \left|\sum_{v=1}^{m} v(t_{vn} - t_{v-1,n})\right| \\ &\leq \frac{1}{m(m+1)} \sum_{v=1}^{m} v\, |\emptyset_{vn}| \\ &= 0(1) \text{ for all } n, m \geq \mathrm{M}\end{aligned}$$

Finally we have

$$\begin{aligned}|g_{mn} - g_{m-1,n}| &= \frac{1}{m(m+1)} \left|\sum_{v=1}^{m} v(d_{vn} - d_{v-1,n})\right| \\ &\leq 0(1) \frac{1}{m(m+1)} \sum v \\ &= 0(1) \text{ for all } n, m \geq M\end{aligned}$$

This implies that

$$\sup_n \sum_{m=1}^{\infty} m^{k-1} |g_{mn} - g_{m-1,n}|^k < \infty$$

Hence

$$P_k \subseteq Q_k \tag{2.8}$$

Combining both results (2.5) and (2.8) we have

$$A_k \subseteq P_k \subseteq Q_k$$

This completes the proof.

Let us consider the following spaces defined earlier for our discussion

$$H_k = \left\{x : \sup_n \sum_{m=1}^{\infty} m^{k-1} |t_{mn} - t_{m-1,n}|^k < \infty\right\}$$
$$B_k = \left\{x : \sup_n \sum_{m=1}^{\infty} m^{k-1} |d_{mn} - d_{m-1,n}|^k < \infty\right\}$$
$$Q_k = \left\{x : \sup_n \sum_{m=1}^{\infty} m^{k-1} |g_{mn} - g_{m-1,n}|^k < \infty\right\} \text{ for } k \geq 1$$

Theorem 2: $H_k \subseteq B_k \subseteq Q_k$

Proof: The inclusion $H_k \subseteq B_k$ has been proved. Now it is the turn to establish the other inclusion that $B_k \subseteq Q_k$.

Let $x \in B_k$. Then by definition

$$\sup_n \sum_{m=1}^{\infty} m^{k-1} |d_{mn} - d_{m-1,n}|^k < \infty$$

Now we consider the case of

$$\sup_n \sum_{m=1}^{\infty} m^{k-1} |g_{mn} - g_{m-1,n}|^k < \infty$$

Since it is known that

$$g_{mn} - g_{m-1,n} = \frac{1}{m(m+1)} \sum_{v=1}^{m} v(d_{vn} - d_{v-1,n})$$

So by Holder's inequality we have

$$|g_{mn} - g_{m-1,n}|^k \leq \frac{1}{m(m+1)^k} \sum_{v=1}^{\infty} v^k\, |d_{vn} - d_{v-1,n}|^k$$

Thus

$$\begin{aligned}\sum_{v=1}^{\infty} m^{k-1} |g_{vn} - g_{v-1,n}|^k &\leq \sum_{v=1}^{\infty} m^{k-1} \left(\frac{1}{m(m+1)^k} \sum_{v=1}^{\infty} v^k\, |d_{vn} - d_{v-1,n}|^k\right) \\ &\leq \sum_{v=1}^{\infty} v^k\, |d_{vn} - d_{v-1,n}|^k \sum_{m=v}^{\infty} \frac{1}{m^2} \\ &\leq \sum_{v=1}^{\infty} v^{k-1}\, |d_{vn} - d_{v-1,n}|^k\end{aligned}$$

From which it follows that
$\sup_n \sum_{m=1}^{\infty} m^{k-1}\left|g_{mn} - g_{m-1,n}\right|^k \leq \sup_n \sum_{m=1}^{\infty} m^{k-1}\left|d_{mn} - d_{m-1,n}\right|^k$

Thus $B_k \subseteq Q_k$ follows and hence

$$H_k \subseteq B_k \subseteq Q_k$$

3. TOPOLOGICAL PROPERTIES

Let us now norm the spaces P_k and Q_k as follows

We define for $x \in P_k$ or $x \in Q_k$ that

$$\left||x|\right|_\mu = \sup_n \left(\sum_{m=1}^{\infty} m^{k-1}\left|g_{mn}(m) - g_{m-1,n}(x)\right|^{1/k}\right)$$

Earlier the spaces A_k and B_k have been proved as Banach Space using the norm.

$$\left||x|\right|_\psi = \sup_n \left(\sum_{m=1}^{\infty} m^{k-1}\left|d_{mn}(m) - d_{m-1,n}(x)\right|^k\right)^{1/k}$$

Now an attempt has been made to prove that P_k and Q_k are Banach spaces by using the norm defined by $\left||x|\right|_\mu$ in the following result.

Theorem 3: The spaces P_k and Q_k are Banach spaces and $\left||x|\right|_\mu \leq \left||x|\right|_\psi$

Proof: Let (x^i) be a Cauchy sequence in P_k .

Then by definition of Cauchy sequence we have

$$\left||x^s - x^t|\right| \to 0 \text{ as } s,t \to \infty$$

$$\begin{aligned}
For\ m = 1,\ g_{mn} - g_{m-1,n} &= g_{1n} - g_{0n} \\
&= \frac{1}{2}\sum_{k=0}^{1} d_{kn} - d_{0n} \\
&= \frac{1}{2}(d_{0n} + d_{1n}) - d_{0n} = \frac{1}{2}(d_{1n} - d_{0n}) \\
&= \frac{1}{2}\left(\frac{1}{2}(t_{0n} + t_{1n}) - t_{0n}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}t_{0n} + \frac{1}{2}t_{1n} - t_{0n}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}t_{1n} - \frac{1}{2}t_{0n}\right) = \frac{1}{4}[t_{1n} - t_{0n}] \\
&= \frac{1}{4}\left[\frac{1}{2}(x_n + x_{n+1}) - x_n\right] = \frac{1}{8}[x_{n+1} - x_n]
\end{aligned}$$

From which it follows that

$$\begin{aligned}
|(g_{1n} - g_{0n})(x^s - x^t)| &= \frac{1}{8}|(x_{n+1}^s - x_{n+1}^t) - (x_n^s - x_{n+1}^t)| \\
&\leq \frac{1}{8}\{|x_{n+1}^s - x_{n+1}^t| + |x_n^s - x_n^t|\} \\
&\leq |x_{n+1}^s - x_{n+1}^t| \leq \left||x^s - x^t|\right| \\
&\to 0 \text{ as } s,t \to \infty
\end{aligned}$$

Hence it follows that (x_n^s) is a Cauchy sequence in C and there exists $x_n \in C$ such that $x_n^s \to x \text{ as } s \to \infty$

Now let us consider the series $x = \sum x_n$ and let us write

$$g_{mn}(x^s - x^t) - g_{m-1,n}(x^s - x^t)$$

as $\qquad (g_{mn} - g_{m-1,n})(x^s - x^t)$

We know that

$$\left||x^s - x^t|\right| = \sup_n \left(\sum_{m=1}^{\infty} m^{k-1}\left|(g_{mn} - g_{m-1,n})(x^s - x^t)\right|^k\right)^{1/k}$$

Therefore, for given $\epsilon > 0$ there exists $s_0 > 0$ such that

$$\sum_{m=1}^{\infty} m^{k-1}\left|(g_{mn} - g_{m-1,n})(x^s - x^t)\right|^k < \epsilon \text{ for all } s,t > s_0$$

Hence for mixed $M > 0$, we have

$$\sum_{m=1}^{M} m^{k-1} \left|\left(g_{mn} - g_{m-1,n}\right)(x^s - x^t)\right|^k < \epsilon \text{ for } s, t > s_0 \text{ for all } n$$

By taking $t \to \infty$ we have

$$\sum_{m=1}^{M} m^{k-1} \left|\left(g_{mn} - g_{m-1,n}\right)(x^s - x^t)\right|^k < \epsilon \text{ for } s > s_0 \text{ for all } n$$

Since $M > 0$ is arbitrary it follows that

$$\sum_{m=1}^{\infty} m^{k-1} \left|\left(g_{mn} - g_{m-1,n}\right)(x^s - x^t)\right|^k < \epsilon \text{ for } s > s_0 \text{ and for all } n$$

Now by taking supremum with respect to n we have

$$\sup_n \sum_{m=1}^{\infty} m^{k-1} \left|\left(g_{mn} - g_{m-1,n}\right)(x^s - x^t)\right|^k < \epsilon \text{ for } s > s_0$$

That is same and saying

$$\left||x^s - x|\right| < \epsilon \text{ for } s > s_0$$

Hence $x^s \to x \ as\ s \to \infty$ on P_k norm and which implies $x \in P_k$

From this it follows that P_k is complete. The proof of the fact that Q_k is complete can be done similarly.

Further, they are normed linear spaces, which can be shown traditionally. Hence they both are Banach spaces. Finally from the discussions made so far we have

$$\sum_{m=1}^{\infty} m^{k-1} \left|g_{mn} - g_{m-1,n}\right|^k \leq \sum_{m=1}^{\infty} m^{k-1} \left|d_{mn} - d_{m-1,n}\right|^k, k \geq 1$$

From which it is evident that

$$\left||x|\right|_{\mu} \leq \left||x|\right|_{\psi}$$

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