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AMICABLE NUMBERS AND GROUPS

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ABSTRACT

In this paper, we extended the notion of amicable numbers to finite groups. Also, we provide some general theorem and present examples of amicable numbers and groups.

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Keywords: *Amicable Numbers, Amicable Groups, Cyclic Groups.*

1. INTRODUCTION

At this time in which mathematical Analysis has opened the way to many profound observations, those problems which have to do with the nature and properties of numbers seem almost completely neglected by Geometers, and the contemplation of numbers has been judged by many to add nothing to Analysis. Yet truly the investigation of the properties of numbers on many occasions requires more acuity than the subtlest questions of geometry, and for this reason it seems improper to neglect arithmetic questions for those.

And indeed the greatest thinkers who are recognized as having made the most important contributions to Analysis have judged the affection of numbers as not unworthy, and in pursuing them have expended much work and study. Namely, it is known that Descartes, even though occupied with the most important meditations on both universal Philosophy and especially Mathematics, spent no little effort uncovering amicable numbers; this matter was then pursued even more by van Schooten.

Let $\sigma(n)$ denote the sum of the divisor of n. Two integers a, b are said to be an amicable (or friendly) pair if $\sigma(a)=\sigma(b)=a+b$. We say an integer n is amicable if it is a member of an amicable pair, or equivalently $\sigma(\sigma(n)-n)=\sigma(n)$. If m=n, they are called perfect numbers, otherwise they form an amicable pair. The first perfect numbers 6, 28, 496. The smallest amicable pair, consisting of the numbers 220 and 284, was known already to the Pythagoreans (ca. 500 BCE) because

$$\begin{cases} \sigma(m)=\sigma(220)=1+2+4+5+10+11+20+22+44+55+110+220=504 \\ \sigma(n)=\sigma(284)=1+2+4+71+142+284=504 \end{cases}. \quad (1)$$

$$\xrightarrow{(1)} \sigma(284)=\sigma(220)=220+284=504.$$

Two further amicable pairs were discovered by medieval Islamic mathematicians, and rediscovered by Fermât and Descartes.

All of these were even numbers. In fact, they were found by the famous rules given by Euclid for perfect, resp. by Thabit ibn Kurrah for amicable numbers (see, e.g., [1], [5] for a survey of this subject), and so were even by construction. L. Euler was the first to study systematically the question whether or not also odd numbers with these properties may be found. The existence of odd perfect numbers has remained a famous open problem in number theory, while the existence of odd amicable numbers was established by Euler. He described several methods to construct numerical examples, one of which is, for example,

A=32×7×13×5 ×17=69615, B=32×7× 13× 107=87633.

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Since Euler's time, many more even and odd amicable pairs have been found and published. A superficial glance at the list of hitherto known odd amicable pairs illustrates the fact that the lack of two as a common factor has to be compensated by a sufficient amount of divisibility by the other small prime factors, like three, five. In fact, all odd amicable pairs that we know [2], [6], [7], [8] actually contain some power of three as a common factor. With some familiarity with the various known methods to find odd amicable pairs, it soon becomes clear, that it is actually very hard to avoid three as a common factor. Paul Bratley and John McKay even conjectured that all odd amicable numbers must be divisible by three, see [3], and also R. Guy's book on open problems in number theory. ([4])

All amicable number pairs below 10^10 have been compiled and published by H. J. J. Te. Riele. ([25]) There are 1427 amicable pairs below 10^10. Subsequently all amicable numbers up to 10^14 have been found. The details of all known amicable pairs can be found there. The distribution of amicable pairs up to 10^14 is given in Table 1.

Table-1

n	Number of amicable Pairs whose smaller number is less than n
10^3	*1*
10^4	*5*
10^5	*13*
10^6	*42*
10^7	*108*
10^8	*236*
10^9	*586*
10^{10}	*1427*
10^{11}	*3340*
10^{12}	*7642*
10^{13}	*17519*
10^{14}	*39374*

These remained the only known amicable numbers for over one thousand years. In the ninth century, the arab mathematician Thabit Ibn Qurra developed a formula for computing pairs of amicable numbers.

Lemma 1.1: The function σ is multiplicative. ([22])

Notice 1.2: For n > 1, let p_n=3× 2^n -1 and q_n=9× $2^{2n-1} - 1$. If p_{n-1}, p_n and q_n are all primes then a= $2^n \times p_{n-1} \times p_n$ and b=$2^n \times q_n$ form a pair of amicable numbers.

His formula produced three pairs of amicable numbers. n = 2 produced the pair a=220, b=284, which were known. n=4 gave the pair a=17, 296, b=18, 416 and n=7 produced the pair a=9, 363, 584, b=9, 437, 056. Evidently, the calculation grew beyond Thabit's ability to continue. In seventeenth century Europe, Thabit's results were not known and in 1636 Fermat calculated the pair 17, 296, 18, 416. Since Fermat and Descartes were rather bitter rivals (some say enemies), Descartes decided that if Fermat found a pair of amicable numbers, he would have to find a pair also. In 1638 Descartes found the pair 9, 363, 584, 9, 437, 056. So almost 2000 years after the first pair of amicable numbers were known only two more pairs were found.

Euler's Rule for amicable pairs) Let n and m are two positive integers with 1≤ m ≤ n −1.

If $\begin{cases} p = 2^n \times (2^{n-m} + 1) - 1 \\ q = 2^m \times (2^{n-m} + 1) - 1 \\ r = 2^{n+m} \times (2^{n-m} + 1)^2 - 1 \end{cases}$ are all primes, then the pair (2^n × p ×q, 2^n × r) is an amicable pair. Note that if n-m =1 in Euler's Rule, we get Thabit's Rule. Even though there are rules to generate amicable numbers, it is not known whether or not there are infinitely many amicable pairs. ([16, 17, 18, 24])

Theorem 1.3: The pair $(2^n \times p \times q, 2^n \times r)$ is amicable pair where $\begin{cases} p = 3 \times 2^{n-1} - 1 \\ q = 3 \times 2^{2n} - 1 \\ r = 9 \times 2^{2n-1} - 1 \end{cases}$ are prime. (n> 1)

Theorem 1.4: (Euler's rule) The pair $(2^n \times p \times q, 2^n \times r)$ is amicable where $\begin{cases} p = 2^m \times (2^{n-m}+1) - 1 \\ q = 2^n \times (2^{n-m}+1) - 1 \\ r = 2^{n+m} \times (2^{n-m}+1) - 1 \end{cases}$ are prime. $(1 \le m \le n)$

Example 1.5: For n = 2, Thabits rule produces the cycle 220, 284. For more ways to compute amicable pairs, see [19].

Table -2: List of amicable numbers from 1 to 20,000,000

The following table, we introduce some amicable pairs.

Table-2: List of amicable numbers from 1 to 20,000,000

a	b	a	b	a	b
220	284	1,328,470	1,483,850	8,619,765	9,627,915
1,184	1,210	1,358,595	1,486,845	8,666,860	10,638,356
2,620	2,924	1,392,368	1,464,592	8,754,130	10,893,230
5,020	5,564	1,466,150	1,747,930	8,826,070	10,043,690
6,232	6,368	1,468,324	1,749,212	9,071,685	9,498,555
10,744	10,856	1,511,930	1,598,470	9,199,496	9,592,504
12,285	14,595	1,669,910	2,062,570	9,206,925	10,791,795
17,296	18,416	1,798,875	1,870,245	9,339,704	9,892,936
63,020	76,084	2,082,464	2,090,656	9,363,584	9,437,056
66,928	66,992	2,236,570	2,429,030	9,478,910	11,049,730
67,095	71,145	2,652,728	2,941,672	9,491,625	10,950,615
69,615	87,633	2,723,792	2,874,064	9,660,950	10,025,290
79,750	88,730	2,728,726	3,077,354	9,773,505	11,791,935
100,485	124,155	2,739,704	2,928,136	10,254,970	10,273,670
122,265	139,815	2,802,416	2,947,216	10,533,296	10,949,704
122,368	123,152	2,803,580	3,716,164	10,572,550	10,854,650
141,664	153,176	3,276,856	3,721,544	10,596,368	11,199,112
142,310	168,730	3,606,850	3,892,670	10,634,085	14,084,763
171,856	176,336	3,786,904	4,300,136	10,992,735	12,070,305
176,272	180,848	3,805,264	4,006,736	11,173,460	13,212,076
185,368	203,432	4,238,984	4,314,616	11,252,648	12,101,272
196,724	202,444	4,246,130	4,488,910	11,498,355	12,024,045
280,540	365,084	4,259,750	4,445,050	11,545,616	12,247,504
308,620	389,924	4,482,765	5,120,595	11,693,290	12,361,622
319,550	430,402	4,532,710	6,135,962	11,905,504	13,337,336
356,408	399,592	4,604,776	5,162,744	12,397,552	13,136,528
437,456	455,344	5,123,090	5,504,110	12,707,704	14,236,136
469,028	486,178	5,147,032	5,843,048	13,671,735	15,877,065
503,056	514,736	5,232,010	5,799,542	13,813,150	14,310,050
522,405	525,915	5,357,625	5,684,679	13,921,528	13,985,672
600,392	669,688	5,385,310	5,812,130	14,311,688	14,718,712
609,928	686,072	5,459,176	5,495,264	14,426,230	18,087,818
624,184	691,256	5,726,072	6,369,928	14,443,730	15,882,670
635,624	712,216	5,730,615	6,088,905	14,654,150	16,817,050
643,336	652,664	5,864,660	7,489,324	15,002,464	15,334,304
667,964	783,556	6,329,416	6,371,384	15,363,832	16,517,768
726,104	796,696	6,377,175	6,680,025	15,938,055	17,308,665
802,725	863,835	6,955,216	7,418,864	16,137,628	16,150,628
879,712	901,424	6,993,610	7,158,710	16,871,582	19,325,698
898,216	980,984	7,275,532	7,471,508	17,041,010	19,150,222
947,835	1,125,765	7,288,930	8,221,598	17,257,695	17,578,785
998,104	1,043,096	7,489,112	7,674,088	17,754,165	19,985,355
1,077,890	1,099,390	7,577,350	8,493,050	17,844,255	19,895,265
1,154,450	1,189,150	7,677,248	7,684,672	17,908,064	18,017,056
1,156,870	1,292,570	7,800,544	7,916,696	18,056,312	18,166,888
1,175,265	1,438,983	7,850,512	8,052,488	18,194,715	22,240,485
1,185,376	1,286,744	8,262,136	8,369,864	18,655,744	19,154,336
1,280,565	1,340,235				

Notice 1.6: The references [13, 14, 15, 20, 23] for further study amicable numbers are introduced.

Now, let G be a finite group. Leinster in [11] extended the notion of perfect numbers to finite groups. He called a finite group is perfect if its order is equal to the sum of the orders of all normal subgroups of the group. In the other words, G is called perfect group if $\sigma(G) = \sum_{N \trianglelefteq G} |N| = 2|G|$. ([12, 21]).

We use this model to describe the definition of amicable groups.

2. AMICABLE NUMBERS AND GROUPS

Proposition 2.1: Let n be a perfect number then the pair of (n, n) is amicable pair.

Corollary 2.2: Let n be a deficient number then the pair of (n, n) is not amicable pair.

Corollary 2.3: Let n be a abundant number then the pair of (n, n) is not amicable pair.

Proposition 2.4: Let n be a natural number and p is a prime where n≠p then the pair of (n, p) is not amicable pair.

Proof: If the pair of (n, p) be an amicable then $\begin{cases}\sigma(p) - p = n \\ \sigma(n) - n = p\end{cases} \to \begin{cases} n = 1 \\ \sigma(n) = p + 1\end{cases} \to \sigma(1) = p + 1.$ This is a contradiction. Therefore, the pair of (n, p) is not amicable pair.

Proposition 2.5: Let m be a perfect number and n be a deficient number then the pair of (m, n) is not amicable pair.
Proof: Let the pair of (m, n) be an amicable, so we have $\begin{cases}\sigma(n) - n = m \\ \sigma(m) - m = n\end{cases}.$ (*)

Since m is perfect number and n is deficient, so $\begin{cases}\sigma(m) = 2m \\ \sigma(n) < 2n\end{cases}$ or $\begin{cases}\sigma(m) - m = m \\ \sigma(n) - n < n\end{cases}.$ (**)

Now, with replacement (**) in (*) we have $\begin{cases}\sigma(n) - n = m < n \\ \sigma(m) - m = n = m\end{cases}.$

But this is a contradiction. Thus the proof is finished.

Corollary 2.6: Similarly, we can show that if that m be a perfect number and n be a abundant number then the pair of (m, n) is not amicable pair.

Proposition 2.7: Let m and n are two deficient numbers then the pair of (m, n) is not amicable pair.

Proof: Let the pair of (m, n) be an amicable pair, so we have

$\begin{cases}\sigma(n) - n = m \\ \sigma(m) - m = n\end{cases}$. By definition m and n, we have $\begin{cases}\begin{cases}\sigma(n) < 2n \\ \sigma(n) - n < n\end{cases} \\ \begin{cases}\sigma(m) < 2m \\ \sigma(m) - m < m\end{cases}\end{cases} \to \begin{cases}\sigma(n) - n = m < n \\ \sigma(m) - m = n < m\end{cases}.$

But this is a contradiction. Thus the proof is finished.

Proposition 2.8: Let σ(n) denote the sum of the divisor of n then σ(n) is a odd number if and only if n be a square or twice the square. ([9])

Proposition 2.9: Let m is an even number and n is an odd number such that (n, m) be the amicable pair. Then n is square.

Proof: According to the assumptions of the theorem, we have $\begin{cases}\sigma(n) - n = m \\ \sigma(m) - m = n\end{cases} \to \begin{cases}\sigma(n) = m + n & \text{is odd} \\ \sigma(m) = n + m & \text{is odd}\end{cases}.$

By using the previous theorem, we have the n is square. Thus the proof is finished.

Theorem 2.10: Let n be a natural number then $\sigma(2^n)$ is an odd number. ([10])

Theorem 2.11: If α be an even number and p be a prime then $\sigma(p^\alpha)$ is an odd number. ([10])

Proposition 2.12: Let m, n are two even natural numbers where $\begin{cases} n = 2^\alpha {p_1}^{a_1} {p_2}^{a_2} \dots {p_k}^{a_k} \\ m = 2^\beta {q_1}^{b_1} {q_2}^{b_2} \dots {q_t}^{b_t}\end{cases}$
where $a_i, b_j \in \mathbb{N}_e$ and $\alpha, \beta \in \mathbb{N}$ then the pair of (m, n) is not amicable.

Proof: According to the assumptions of the theorem, we have
$$\begin{cases}\sigma(n)=\sigma(2^{\alpha}p_1^{a_1}p_2^{a_2}\dots p_k^{a_k})=\sigma(2^{\alpha})\sigma(p_1^{a_1})\dots\sigma(p_k^{a_k})\\ \sigma(m)=\sigma(2^{\beta}q_1^{b_1}q_2^{b_2}\dots q_t^{b_t})=\sigma(2^{\beta})\sigma(q_1^{b_1})\dots\sigma(q_t^{b_t})\end{cases}.$$

By using the previous theorems we have $\begin{cases}\sigma(n)=\sigma(2^{\alpha}p_1^{a_1}p_2^{a_2}\dots p_k^{a_k})=\sigma(2^{\alpha})\sigma(p_1^{a_1})\dots\sigma(p_k^{a_k}) & \text{is odd}\\ \sigma(m)=\sigma(2^{\beta}q_1^{b_1}q_2^{b_2}\dots q_t^{b_t})=\sigma(2^{\beta})\sigma(q_1^{b_1})\dots\sigma(q_t^{b_t}) & \text{is odd}\end{cases}$.

Therefore, $\begin{cases}\sigma(n)\neq m+n\\ \sigma(m)\neq m+n\end{cases}$. Hence, the proof is finished.

Proposition 2.13: Let m, n are two odd natural numbers where $\begin{cases}n=p_1^{a_1}p_2^{a_2}\dots p_k^{a_k}\\ m=q_1^{b_1}q_2^{b_2}\dots q_t^{b_t}\end{cases}$ where $a_i, b_j\in\mathbb{N}_e$, then the pair of (m, n) is not amicable.

Proof: According to the assumptions of the theorem, we have $\begin{cases}\sigma(n)=\sigma(p_1^{a_1}p_2^{a_2}\dots p_k^{a_k})=\sigma(p_1^{a_1})\dots\sigma(p_k^{a_k})\\ \sigma(m)=\sigma(q_1^{b_1}q_2^{b_2}\dots q_t^{b_t})=\sigma(q_1^{b_1})\dots\sigma(q_t^{b_t})\end{cases}$.

By using the previous theorems we have $\begin{cases}\sigma(n)=\sigma(p_1^{a_1}p_2^{a_2}\dots p_k^{a_k})=\sigma(p_1^{a_1})\dots\sigma(p_k^{a_k}) & \text{is odd}\\ \sigma(m)=\sigma(q_1^{b_1}q_2^{b_2}\dots q_t^{b_t})=\sigma(q_1^{b_1})\dots\sigma(q_t^{b_t}) & \text{is odd}\end{cases}$. Therefore, $\begin{cases}\sigma(n)\neq m+n\\ \sigma(m)\neq m+n\end{cases}$. Therefore, the proof is finished.

Proposition 2.14: Let p be a prime and n be a natural number where n$\neq$ p^2. Then the pair of (n, p^2) is not amicable pair.

Proof: If the pair of (n, p^2) is amicable then $\begin{cases}\sigma(n)-n=p^2\\ \sigma(p^2)-p^2=n\end{cases}\rightarrow\begin{cases}\sigma(n)-n=p^2\\ 1+p+p^2-p^2=n\end{cases}\rightarrow\begin{cases}\sigma(n)-n=p^2\\ 1+p=n\end{cases}$. The resulting solution is a contradiction because $\sigma(p+1)=1+p+p^2=\sigma(p^2)$. Therefore, the pair of (n, p^2) is not amicable pair.

Definition 2.15: (Extension of the Definition of Amicable Numbers) The numbers n_1, $n_2,\dots,n_k$ are called k-amicable if $\sigma(n_1)=\sigma(n_2)=\dots=\sigma(n_k)=n_1+n_2+\dots+n_k$. For example, triplex ($2^5\times3^3\times47\times109$, $2^5\times3^2\times7\times659$, $2^5\times3^2\times5279$) is a 3-amicable numbers because
$$\sigma(2^5\times3^3\times47\times109)=\sigma(2^5\times3^2\times7\times659)=\sigma(2^5\times3^2\times5279)$$
$$=2^5\times3^3\times47\times109+2^5\times3^2\times7\times659+2^5\times3^2\times5279.$$

Definition 2.16: Let G_1 and G_2 be finite groups. Then the pair of (G_1, G_2) is called amicable groups if $\sigma(G_1)=\sigma(G_2)=|G_1|+|G_2|$.

Example 2.17: The smallest pair of amicable groups is (C_{220}, C_{284}) because $\begin{cases}\sigma(C_{220})-|C_{220}|=|C_{284}|\\ \sigma(C_{284})-|C_{284}|=|C_{220}|\end{cases}\rightarrow$
$$\begin{cases}\sigma(220)-|220|=|284|\\ \sigma(284)-|284|=|220|\end{cases}\rightarrow\begin{cases}\sigma(220)=\sigma(4\times71)=504=284+220=504\\ \sigma(284)=\sigma(4\times5\times11)=504=284+220=504\end{cases}.$$

Example 2.18: Let C_n be the cyclic group of order n and p be a prime then the pair of (C_{17296}, C_{18416}) is amicable groups. Because
$$\begin{cases}\sigma(C_{17296})-|C_{17296}|=|C_{18416}|\\ \sigma(C_{18416})-|C_{18416}|=|C_{17296}|\end{cases}\rightarrow\begin{cases}\sigma(17296)-17296=18416\\ \sigma(18416)-18416=17296\end{cases}$$
$$\rightarrow\begin{cases}\sigma(17296)=\sigma(16\times23\times47)=35712=17296+18416=35712\\ \sigma(18416)=\sigma(16\times1152)=35712=17296+18416=35712\end{cases}.$$

Definition 2.19: (Extension of the Definition of Amicable Groups) Let G_1, $G_2,\dots,G_k$ be finite groups then G_1, $G_2,\dots,G_k$ are called k-amicable if $\sigma(G_1)=\sigma(G_2)=\dots=\sigma(G_k)=|G_1|+|G_2|+\dots+|G_{k-1}|+|G_k|$. For example, triplex ($C_{2^5\times3^3\times47\times109}$, $C_{2^5\times3^2\times7\times659}$, $C_{2^5\times3^2\times5279}$) is a 3-amicable groups because $\sigma(C_n)=\sigma(n)$.

Proposition 2.20: Let C_n be the cyclic group of order n and p be a prime. Then the pair of (C_p, C_{p^2}) is not amicable groups.

Proof: If the pair of (C_p, C_{p^2}) is amicable groups then
$\begin{cases}\sigma(C_p)-|C_p|=|C_{p^2}|\\ \sigma(C_{p^2})-|C_{p^2}|=|C_p|\end{cases}\rightarrow\begin{cases}\sigma(p)-p=p^2\\ \sigma(p^2)-p^2=p\end{cases}\rightarrow\begin{cases}1+p-p=p^2\\ 1+p+p^2-p^2=p\end{cases}\rightarrow\begin{cases}1=p^2\\ 1=0\end{cases}$. The resulting solution is a contradiction. Therefore, the pair of (C_p, C_{p^2}) is not amicable groups.

Proposition 2.21: Let m, n are natural numbers and C_n be the cyclic group of order n then the pair of (m, n) is a amicable ↔ the pair of (C_m, C_n) is amicable.

Proof: The proof of the theorem is obvious because $\sigma(C_n) = \sigma(n)$.

Corollary 2.22: Let G_1 and G_2 are two finite groups whose $G_1 \sim G_2$ then we know that $\sigma(G_1) = \sigma(G_2)$. Therefore, we can say that the number of amicable groups is greater than (or equal) the number of amicable numbers.

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