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A REPRESENTATION FOR MINIMUM CONGRUENCES IN LATTICES

C. Ganesa Moorthy[1] and SG. Karpagavalli*[2]

[1]*Department of Mathematics, Alagappa University, Karaikudi-630 004, Tamil Nadu, India.*

[2]*Department of Mathematics,*
Dr.Umayal Ramanathan college for women, Karaikudi-630 004, Tamil Nadu, India.

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ABSTRACT

The least upper bound and the greatest lower bound of a collection of congruences in a lattice can be realized as a quotient of direct limit and as a sub lattice of inverse limit. This is explained in this article.

Key words: *Inverse limit, Congruence relation, Refinement.*

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1. INTRODUCTION

A poset (partially ordered set) is a set P with a partial order relation ≤ which is reflexive, anti-symmetric and transitive. A lattice is a poset (P, ≤) in which any two elements have a least upper bound and a greatest lower bound. An equivalence relation is a relation that is reflexive, symmetric and transitive. All equivalence classes of an equivalence relation form a partition; and a partition leads to an equivalence relation. If two partitions P_1 and P_2 of a set X are such that P_1 is a refinement of P_2 then it is written as $P_1 \leq P_2$. This relation makes the collection of all partitions as a complete lattice in which every subset has a least upper bound and a greatest lower bound. An equivalence relation θ on a set X is sometimes used in the following form: $x \equiv y \pmod{\theta}$, when x and y are related by θ in X. An equivalence relation θ on a lattice (L, ≤) or (L, ∨, ∧) is called a congruence relation, if it has the following substitution properties: $x \vee z \equiv y \vee z \pmod{\theta}$ and $x \wedge z \equiv y \wedge z \pmod{\theta}$, whenever $x \equiv y \pmod{\theta}$, and $x, y, z \in L$. The collection of all congruences on a lattice L form a sub lattice of the lattice of all partitions of L and it is also a complete lattice. It is known (see Theorem 3.9 in [1]) how to construct the least upper bound and the greatest lower bound of a given collection of congruences. A construction for the same in terms of inverse limit and direct limit is explained here. It is expected that every view on congruence lattices would be helpful to understand the structure of congruence lattices.

2. DEFINITIONS

Let us say that a subset A of a lattice (L, ∨, ∧) is closed in L
(i) if A contains least upper bound of any subset of A whenever it exists in L, and
(ii) if A contains greatest lower bound of any subset of A whenever it exists in L.

Let us further say that an equivalence relation θ is closed in (L, ∨, ∧), if each equivalence class of θ is a closed subset of L. A poset is a directed set if any two elements have an upper bound. A poset is an inversely directed set if any two elements have a lower bound. Different books follow different terminologies.

Definition 2.1: Let (D, ≤) be a directed set. A family $\{f_{ij} : X_j \to X_i : i, j \in D, i \leq j\}$ of functions along with a family $(X_i)_{i \in D}$ of sets is called an inverse system, if for $i \leq j \leq k$ in D, we have $f_{ij} \circ f_{jk} = f_{ik}$. The inverse limit of this inverse system is the subset $\{(x_k)_{k\in D} \in \prod_{k\in D} X_k : f_{ij}(x_j) = x_i$, whenever $i \leq j$ in $D\}$ of the Cartesian product $\prod_{k\in D} X_k$.

**Corresponding author: SG. Karpagavalli*[2]*

Definition 2.2: Let $(D, \leq)$ be an inversely directed set. A family $\{f_{ij} : X_j \to X_i : i, j \in D, i \leq j\}$ of functions along with a family $(X_i)_{i\in D}$ of sets is again called an inverse system, if for $i \leq j \leq k$ in D, we have $f_{ij} \circ f_{jk} = f_{ik}$. The direct limit of this inverse system is the subset $\{(x_k)_{k\in D}: \Pi_{k\in D}X_k: f_{ij}(x_j) = x_i$, whenever $i \leq j$ in D$\}$ of the Cartesian product $\Pi_{k\in D}X_k$. Let us define the direct co finite limit as $\{(x_k)_{k\in D} \in \Pi_{k\in D} X_k :$ there is a $k \in D$ such that $f_{ij}(x_j) = x_i$, whenever $i \leq j \leq k$ in D $\}$. Let us define an equivalence relation '~' on this direct co finite limit by $(x_k)_{k\in D} \sim (y_k)_{k\in D}$ if there is a $k \in D$ such that $x_i = y_i$ for $i \leq k$. Then the direct co finite limit is defined as the collection of all equivalence classes. Let us recall that the product $\leq$ on $\Pi_{i\in D}X_i$ for a given collection of lattices $(X_i, \leq_i)$ or $(X_i, \vee_i, \wedge_i)$, $i \in D$, is defined by the relation $(x_k)_{k\in D} \leq (y_k)_{k\in D}$ if and only if $x_i \leq y_i$, $\forall\, i\in D$. Note that the 'join' in the product lattice satisfies the relation $(x_k)_{k\in D} \vee (y_k)_{k\in D} = (x_k \vee y_k)_{k\in D}$, and the meet in the product lattice satisfies the relation $(x_k)_{k\in D} \wedge (y_k)_{k\in D} = (x_k \wedge y_k)_{k\in D}$.

3. FUNDAMENTAL LEMMAS

Lemma 3.1: Let us assume further in definition 2.1 that each X_i is a lattice $(X_i, \leq_i)$ or $(X_i, \vee_i, \wedge_i)$. Let each f_{ij} be a lattice homomorphism. Then the inverse limit is a sub lattice of the product lattice $\Pi_{i\in D} X_i$.

Proof: Note that, for given $(x_k)_{k\in D}$, $(y_k)_{k\in D}$ in the inverse limit, we have $f_{ij}(x_j \vee_j y_j) = x_i \vee_i y_i$, whenever $i \leq j$ in D so that $(x_k)_{k\in D} \vee (y_k)_{k\in D}$ is in the inverse limit. Similarly the inverse limit is also closed under 'meet'.

Lemma 3.2: Let us assume further in definition 2.2 that each X_i is a lattice $(X_i, \leq_i)$ or $(X_i, \vee_i, \wedge_i)$. Let each f_{ij} be a lattice homomorphism. Then the direct limit is a sub lattice of the product lattice $\Pi_{i\in D} X_i$, and the direct co finite limit is also a sub lattice of the product lattice.

Proof: It is possible as in the proof of the lemma 3.1 to prove that the direct limit is a sub lattice of the product lattice. Let $(x_k)_{k\in D}$ and $(y_k)_{k\in D}$ be in the direct co finite limit. Since D is inversely directed, there is a $k\in D$ such that $f_{ij}(x_j) = x_i$ and $f_{ij}(y_j) = y_i$, whenever $i \leq j \leq k$ in D. In this case, we have, $f_{ij}(x_j \vee_j y_j) = x_i \vee_i y_i$ and $f_{ij}(x_j \wedge_j y_j) = x_i \wedge_i y_i$, whenever $i \leq j \leq k$ in D. Thus $(x_k)_{k\in D} \vee (y_k)_{k\in D}$ and $(x_k)_{k\in D} \wedge (y_k)_{k\in D}$ are in the direct co finite limit. This completes the proof.

Lemma 3.3: Let D, X_i, and f_{ij} be as in the statement of lemma 3.2. Consider the equivalence relatioin ' ~' given in the definition 2.2. Then ' ~' is a congruence relation so that direct quotient limit becomes a lattice.

Proof: Suppose $(x_k)_{k\in D} \sim (y_k)_{k\in D}$ in the direct co finite limit and $(z_k)_{k\in D}$ be in thedirect co finite limit. Then there is a k in the inversely directed set D such that $x_i = y_i$ for $i \leq k$ in D. Then $x_i \vee_i z_i = y_i \vee_i z_i$ and $x_i \wedge_i z_i = y_i \wedge_i z_i$ for $i \leq k$ in D. This proves that $(x_k)_{k\in D} \vee (z_k)_{k\in D} \sim (y_k)_{k\in D} \vee (z_k)_{k\in D}$ and $(x_k)_{k\in D} \wedge (z_k)_{k\in D} \sim (y_k)_{k\in D} \wedge (z_k)_{k\in D}$.

This completes the proof.

4. MAIN THEOREMS

Let $(X, \vee, \wedge)$ be a given lattice with a collection of congruence relations $(\theta_i)_{i\in D}$, and let $((X_i, \vee_i, \wedge_i))_{i\in D}$ be the collection of lattices $X_i = X/\theta_i$. To each $i \in D$, let $T_i : X \to X_i$ be the quotient mapping which is a surjective lattice homomorphism. To each $i \in D$, the partition $P_i = \{T_i^{-1}(a_i) : a_i \in X_i\}$ corresponds to the congruence relation θ_i. Let us consider the usual (refinement) order relation in the complete lattice of all partitions on X. Let $\wedge_{i\in D}\, \theta_i$ and $\vee_{i\in D}\, \theta_i$ denote the congruence relations corresponding to infimum and supremum of the partitions $(P_i)_{i\in D}$ of $(\theta_i)_{i\in D}$ respectively. Let us follow these notations in the next two theorems 4.1 and 4.2.

Theorem 4.1: Let D be a directed set such that $\theta_j \leq \theta_i$, whenever $i \leq j$ in D. Let $f_{ij} : X_j \to X_i$ be the natural lattice homomorphism such that $f_{ij} \circ T_j = T_i$, whenever $i \leq j$ in D. Let us further assume that X is complete and each θ_i is a closed congruence relation. Then the lattice of inverse limit discussed in lemma 3.1 is lattice isomorphic with $X / \wedge_{i\in D}\theta_i$. Moreover $\wedge_{i\in D}\, \theta_i$ is a closed relation.

Proof: The hypotheses of lemma 3.1 are satisfied and hence the inverse limit lattice exists. Let $(Y, \vee, \wedge)$ be the inverse limit. Define $T: X \rightarrow Y$ by $T(x) = (T_i(x))_{i\in D}, \forall x\in X$. Then, by definitions, T is a lattice homomorphism that maps X into Y. Let $(x_k)_{k\in D}\in Y \subseteq \Pi_{k\in D} X_k$. Then $\cap_{k\in D} T^{-1}_k(x_k)$ is non empty, because $T^{-1}_i(x_i)\supseteq T^{-1}_j(x_j)$,whenever $i\leq j$ in the directed set D, and because X is complete and each θ_i is a closed congruence relation. Infact, $\inf_{k\in D} \sup_{i\geq k} y_i$ is a member of this intersection, when $y_i\in T^{-1}(x_i)$. Also if $z\in\cap_{k\in D} T^{-1}_k(x_k)$, then $T_i(z)=x_i$, $\forall i\in D$. Thus T is surjective. If $x\in A\cap(\cap_{k\in D}T^{-1}_k(x_k))$ and $y\in\cap(\cap_{k\in D} T^{-1}_k(x_k))$ for two distinct equivalence classes A and B for the equivalence relation $\wedge_{i\in D} \theta_i$, then there at least one k in D such that $x\equiv y \pmod{\theta_k}$. Thus $T_k(x)\neq T_k(y)$, whereas $T_k(x)=x_k, k\in D$, then we find that $A=\cap_{k\in D}T^{-1}_k(x_k)$. Thus each member of the collection of all equivalence classes corresponding to $\wedge_{i\in D} \theta_i$ is mapped by T onto a unique member of Y. This proves the theorem, because $\cap_{k\in D} T^{-1}_k(x_k)$ is a closed subset of X, for each $x\in X$.

If the additional conditions that X is complete and each θ_i is closed are relaxed, then, $X/\wedge_{i\in D} \theta_i$ can be realized as a sub lattice of the inverse limit induced by (X_i) and (f_{ij}). This observation follows from the previous proof.

Theorem 4.2: Let D be an inversely directed set such that $\theta_j\leq\theta_i$, whenever $i\leq j$ in D. Let $f_{ij}:X_j\rightarrow X_i$ be the natural lattice homomorphism such that $f_{ij}\circ T_j=T_i$, whenever $i\leq j$ in D. Then the lattice of direct quotient limit discussed in lemma 3.3 is lattice isomorphic with $X/\vee_{i\in D} \theta_i$.

Proof: The hypotheses of lemma 3.3 are satisfied and hence the direct limit lattice and the direct quotient limit lattice exist. Let Y and Z denote the direct co finite limit and the direct quotient limit respectively, and let us also follow the notations used in lemma 3.2 and lemma 3.3. Define $T: X\rightarrow Y$ by $T(x)=(T_i(x))_{i\in D}$, $\forall x\in X$. Define $S:Y\rightarrow Z$ by $S((x_k)_{k\in D})= [(x_k)_{k\in D}]$, the equivalence class defined by $(x_k)_{k\in D}$ corresponding to the relation '~', $\forall (x_k)_{k\in D}\in Y$. Then by definitions T and S are lattice homomorphisms. Let $[(x_i)_{i\in D}]\in Z$, when $(x_i)_{i\in D}\in Y$ Then there is an element $x\in X$ and there is a $k_x\in D$ such that $T_i(x)=x_i$, for $i\leq k_x$ in D. If $(y_i)_{i\in D} \in Y\cap[(x_i)_{i\in D}]$. Then there is a $k\leq k_x$ in D such that $y_i=x_i$, $\forall i\leq k$ in D, and hence to any $z\in T^{-1}_k(x_k)$, we have $(S\circ T)(z)=[(x_i)_{i\in D}]=[(y_i)_{i\in D}]$. This, of course, verifies that $x\in(S\circ T)^{-1}[(x_i)_{i\in D}]\subseteq\cap_{k\in D}\cup_{i\leq k, i\in D}T^{-1}_i(x_i)$. In particular $S\circ T: X\rightarrow Z$ is surjective. The remaining part of this paragraph is written to record additional information that is not required for proof. If $y\in\cap_{k\in D}\cup_{i\leq k, i\in D} T^{-1}_i(x_i)$, then $y\in\cup_{i\in k}, T^{-1}_i(x_i)$, $\forall k\in D$. Thus, for every $k\in D$ there is a $k_y\leq k$ in D such that $k_y \leq k_x$ and $T_i(y)=x_i=T_i(x)$ for $i\leq k_y$, because $T^{-1}_j(x_j)\subseteq T^{-1}_i(x_i)$, whenever $i\leq j\leq k_x$ in D. Thus $y\in(S\circ T)^{-1}[(x_i)_{i\in D}]$. So, we have an additional information that $(S\circ T)^{-1}[(x_i)_{i\in D}]= \cap_{k\in D}\cup_{i\in k} T^{-1}_i(x_i)$. Now let x, y belong to an equivalence class defined by $\vee_{i\in D} \theta_i$. Then there are $\theta i_1, \theta i_2,..., \theta i_n$ with $i_j \in D$ and there are points $x_1,x_2,...,x_{n-1}$ in X such that $x\equiv x_1 \pmod{\theta_1}$, $x_1\equiv x_2 \pmod{\theta_2}$, $x_2\equiv x_3 \pmod{\theta_3}$,... $x_{n-2}\equiv x_{n-1} \pmod{\theta_{n-1}}$, $x_{n-1}\equiv y \pmod{\theta_n}$ (see the proof of theorem3.9 in [1]). Find $k\in D$ such that $k\leq i_j$, for j=1,2,...,n. Then $x\equiv y \pmod{\theta_k}$, and hence $(S\circ T)(x)=((S\circ T)(y)$. On the other hand if $(S\circ T)(x)=((S\circ T)(y)$, then there is a $k\in D$ such that $x\equiv y \pmod{\theta_k}$ and hence $x\equiv y \pmod{\vee_{i\in D} \theta_i}$. So $(S\circ T)^{-1}[(x_i)_{i\in D}]$ is precisely one equivalence class defined by $\vee_{i\in D} \theta_i$ and every equivalence class defined by $\vee_{i\in D} \theta_i$ should be of the form $(S\circ T)^{-1}[(x_i)_{i\in D}]$. This completes the proof of the theorem.

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