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A FIXED POINT THEOREM IN G-METRIC SPACE

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ABSTRACT

In this paper we prove a common fixed theorem in G-metric space using pairs of weakly compatible mappings.

Key Words: *Complete G-metric space, weakly compatible mapping,*

INTRODUCTION

Banach contraction principle has been generalized in various spaces through different mappings. It has been a centre of rigorous research. After Gahler gave the concept of 2-metric space Dhage [2, 3] introduced the concept of D-metric space, but most of the results in D-metric space were proven invalid by Mustafa and Sims [14, 15]. They further introduced the concept of G-metric. Here we prove a common fixed point theorem in G-metric space, for six pairs of weakly compatible mappings.

DEFINITIONS AND PRELIMINARIES

We here begin with some definitions and results for G- metric spaces that will be used in the following sections.

Definition 2.1: [15] Let X be a nonempty set. and let G; X x X x X--- >R^+ be a function satisfying the following axioms

(G_1) G (x, y, z) = 0 if x = y = z
(G_2) G (x, x, y) > 0, for all x, y ε X with x ≠ y
(G_3) G (x, x, y) ≤ G (x, y, z) for all x, y, z ε X with z ≠ y .
(G_4) G (x, y, z) = G (x, z, y) = G (y, z, x) =….. (Symmetry in all three variables)
(G_5) G (x, y, z) ≤ G (x, a, a) + G (a, y, z), for all x, y, z, a ε x (rectangle inequality)

Then the function G is called a generalized metric or more specifically a G- metric on X, and the pair (X, G) is called a G- metric space .

Definition 2.2: [15] Let (X, G) be a G- metric space, let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ converges to a point x in X

if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$

In other words for e ε> 0 there exists n_oε N such that G (x, x_n, x_m) <ε for all n, m ≥ n_o Then x is called the limit of sequence $\{x_n\}$.

Definition 2.3: [15] Let (X, G) be a G- metric space, a sequence $\{x_n,\}$ is called G - Cauchy sequence if for given ε > o, there is n_oε N such that

G (x_n, x_m, x_e) < ε for all n, m, l ≥ n_o that is if. G (x_n , x_m, x_e) → 0 as n, m, $l\to\infty$

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Preposition 2.5: [15] Let (X, G) be a G-metric space, Then, the following are equivalent

(i) $\{x_n\}$is G- convergent to x
(ii) $G(x_n, x_n, x) \to 0$, as $n \to\infty$
(iii) $G(x_n, x, x,) \to 0$, as $n \to\infty$
(iv) $G(x_m, x_n, x) \to 0$ as $n, m \to\infty$

Preposition 2.6: [15] In a G-metric space (X, G) the following are equivalent

(i) The sequence $\{x_n\}$ is G- Cauchy
(ii) For every $\varepsilon > 0$, there exists $n_o \varepsilon N$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq n_0$.

Definition 2.7: [16] Let ϕ denote the set of alternating distance functions $\phi : [0, \phi\ [\to [0, \infty[$which satisfies following conditions

(i) ϕ is strictly increasing
(ii) ϕ is upper semi continuous from the right.
(iii) $\sum_{n=0}^{\infty} \phi(t) <\infty$ for all $t > 0$
(iv) $\phi(t) = 0 \Leftrightarrow t = 0$

MAIN RESULT

Let f, g, h, r, s, and t be self mappings of a complete G-metric space (X, G) and

(i) $f(X)\subseteq t(X)$, $g(X)\subseteq s(X)$, $h(X)\subseteq r(X)$ and f (X) or g (X) or h (X) is a closed subset of X.
(ii) $G(fx, gy, hz) \leq\phi\ \{ \max \{ G(gy, fx, rx), G(hz, gy, ty,),G(fx, sz, hz), \alpha G(fx, rx, gy) + \gamma\ G(sz, fx, rx), \beta G(gy, ty, hz)+ \delta G(fx, gy, ty)\}\}$where $\alpha, \beta, \gamma, \delta, \geq 0$,$\alpha + \beta + \gamma + \delta< 1/2$
(iii) $\phi : R^+\to R^+$ is increasing function such that $\phi(a) < a$ for all $a > 0$ and $\sum\phi(a) <\infty$
(iv) The pairs (f, r), (g, t) and (h, s) are weakly compatible pairs of mappings. Then the mappings f, g, h, r, s and t have a unique common fixed point.

Proof: Let $x_0\in X$ be an arbitrary point. Then from (i) there exists $x_1, x_2, x_3\in X$ such that $fx_0 = tx_1 = y_0$, $gx_1 = sx_2 = y_1$ and $hx_2 = rx_3 = y_2$ inductively we define a sequence $\{y_n\}$ in X such that $fx_{3n} = tx_{3n+1} = y_{3n}$, $gx_{3n+1} = sx_{3n+2+} = y_{3n+1}$ and $hx_{3n+2} = rx_{3n+3}= y_{3n+2}$ for n= 0, 1, 2 .....

We now prove that $\{y_n\}$ is a Cauchy sequence and for this we define
$d_m = G(y_m , y_{m+1}, y_{m+2})$. so we have.

$$d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2})$$

$$= G(fx_{3n}, gx_{3n+1}, hx_{3n+2})$$

$$\leq\phi \{\max \{G(gx_{3n+1}, fx_{3n}, rx_{3n}), G(hx_{3n+2}, gx_{3n+1}, tx_{3n+1}),G(fx_{3n}, sx_{3n+2}, hx_{3n+2}), \alpha G(fx_{3n}, rx_{3n}, gx_{3n+1}), + \gamma\ G(sx_{3n+2}, fx_{3n}, rx_{3n}),\beta G(gx_{3n+1}, tx_{3n+1}, hx_{3n+2}) + \delta\ G(fx_{3n}, gx_{3n+1}, tx_{3n+1})\}\}$$

$$\leq\phi \{\max \{G(y_{3n+1}, y_{3n}, y_{3n-1}), G(y_{3n+2}, y_{3n+1}, y_{3n}),G(y_{3n}, y_{3n+1}, y_{3n+2}),\alpha G(y_{3n}, y_{3n-1}, y_{3n+1}), +\gamma\ G(y_{3n+1}, y_{3n}, y_{3n-1}),\beta G(y_{3n+1}, y_{3n}, y_{3n+2}) + \delta\ G(y_{3n}, y_{3n+1}, y_{3n})\}\}$$

$$\leq\phi \{\max \{d_{3n-1}, d_{3n}, d_{3n}, \alpha\ d_{3n-1}+\gamma\ d_{3n-1},\beta\ d_{3n}+ \delta\ d_{3n}\} \text{ as } G(a, a, x) \leq G(x, y, z)$$

$$\leq\phi \{\max \{d_{3n-1}, (\gamma + \alpha)\ d_{3n-1}, (\beta + \delta)\ d_{3n}\}$$

From the above inequality we have following cases

Case-I: If max = d_{3n-1} then from the inequality

$$d_{3n} \leq \phi\ \{d_{3n-1}\}\leq d_{3n-1} \text{ as } \phi(a) < a \text{ for all } a > 0 .$$

Case-II: $d_{3n}\leq \phi\ \{d_{3n}\} < d_{3n}$ which is a contradiction.

Case-III: If max = $(\alpha + \gamma)\ d_{3n-1}$ then from the inequality

$$d_{3n} \leq \phi\ \{(\alpha +\gamma)\ d_{3n-1}\} < (\alpha + \gamma)\ d_{3n-1}$$
$$d_{3n} < d_{3n-1}$$

Case-IV: If max = $(\beta + \delta)\, d_{3n}$, then from the inequality we have

$$d_{3n} \leq \phi\{(\beta + \delta)\, d_{3n}\} < (\beta + \delta)\, d_{3n}$$

$d_{3n} < d_{3n}$ which is a contradiction. Hence in either case we infer that $d_{3n} \leq d_{3n-1.}$

Consider,

$d_{3n+1} = G\,(y_{n+1}, y_{n+2}, y_{n+3},)$

$\leq G\,(fx_{3n+1}, gx_{3n+2}, hx_{3n+3},)$

$\leq \phi\{\max\{G(gx_{3n+2}, fx_{3n+1}, rx_{3n+1},), G(hx_{3n+3}, gx_{3n+2}, tx_{3n+2}), G(fx_{3n+1}, sx_{3n+3}, hx_{3n+3}),$

$\alpha G(fx_{3n+1}, rx_{3n+1}, gx_{3n+2}) + \gamma\, G\,(sx_{3n+3}, fx_{3n+1}, rx_{3n+1}), \beta G\,(gx_{3n+2}, tx_{3n+2}, hx_{3n+3}) + \delta\, G\,(fx_{3n+1}, gx_{3n+2}, tx_{3n+2})\}\}$

$\leq \phi\,\{\max\{G\,(y_{3n+2}, y_{3n+1}, y_{3n}),\; G\,(y_{3n+3}, y_{3n+2}, y_{3n+1}), G\,(y_{3n+1}, y_{3n+2}, y_{3n+3}), \alpha\, G\,(y_{3n+1}, y_{3n}, y_{3n+2})$
$+\gamma\, G\,(y_{3n+2}, y_{3n+1}, y_{3n}),\; \beta\, G\,(y_{3n+2}, y_{3n+1}, y_{3n+3}) + \delta\, G\,(y_{3n+1}, y_{3n+2}, y_{3n+1})\}\}$

$\leq \phi\,\{\max\{d_{3n}, d_{3n+1}, d_{3n+1},\; \alpha\, d_{3n},\; + \gamma\, d_{3n},\; \beta\, d_{3n+1} + \delta\, d_{3n+1}\}\}$ as $G\,(a, a, x) \leq G\,(x, y, z)$

$\leq \phi\,\{\max\{d_{3n}, d_{3n+1}, (\alpha + \gamma)\, d_{3n}, (\beta + \delta)\, d_{3n+1}\}$

We have following cases

Case-I: max = d_{3n} then from above inequality $d_{3n+1} \leq \phi\,(d_{3n}) < d_{3n}$ as $\phi\,(a) < a$ for all $a > 0$

Case-II: max = d_{3n+1} then we have ,$d_{3n+1} \leq \phi\,(d_{3n+1}) < d_{3n+1}$ which is a contradiction.

Case-III: max = $(\alpha + \gamma)\, d_{3n}$ then we have .
$d_{3n+1} \leq \phi\{(\alpha + \gamma)\, d_{3n}\} < (\alpha + \gamma)\, d_{3n}$, as $\alpha + \beta + \gamma + \delta < 1/2$ we have $d_{3n+1} \leq d_{3n}$

Case-IV: max = $(\beta + \delta)\, d_{3n+1}$ then from the inequality.
$d_{3n+1} \leq \phi\,(\beta + \delta)\, d_{3n+1}\} < (\beta + \delta)\, d_{3n+1}$,as $\alpha + \beta + \gamma + \delta < 1/2$, $d_{3n+1} < d_{3n+1}$ is a contradiction

Hence in either case we have $d_{3n+1} \leq d_{3n}$ Now consider.
$d_{3n+2} = G\,(y_{3n+2}, y_{3n+3}, y_{3n+4})$

$\leq G\,(fx_{3n+2}, gx_{3n+3}, hx_{3n+4})$

$\leq \phi\,\{\max\{G\,(g\,x_{3n+3}, fx_{3n+2}, rx_{3n+2},), G(hx_{3n+4}, gx_{3n+3}, tx_{3n+3}), G\,(fx_{3n+2}, sx_{3n+4}, hx_{3n+4}),$
$\alpha\, G\,(fx_{3n+2}, rx_{3n+2}, gx_{3n+3},) + \gamma\, G(sx_{3n+4}, fx_{3n+2}, rx_{3n+2}), \beta\, G\,(gx_{3n+3}, tx_{3n+3}, hx_{3n+4}) + \delta\, G\,(fx_{3n+2}, gx_{3n+3}, tx_{3n+3})\}\}$

$\leq \phi\,\{\max\{G\,(y_{3n+3}, y_{3n+2}, y_{3n+1}),\; G\,(y_{3n+4}, y_{3n+3}, y_{3n+2}), G\,(y_{3n+2}, y_{3n+3}, y_{3n+4}), \alpha\, G\,(y_{3n+2}, y_{3n+1}, y_{3n+3})$
$+\gamma\, G\,(y_{3n+3}, y_{3n+2}, y_{3n+1}), \beta\, G\,(y_{3n+3}, y_{3n+2}, y_{3n+4}) + \delta\, G\,(y_{3n+2}, y_{3n+3}, y_{3n+2})\}\}$

$\leq \phi\,\{\max\{d_{3n+1}, d_{3n+2}, d_{3n+2},\; \alpha\, d_{3n+1}, + \gamma\, d_{3n+1},\; \beta\, d_{3n+2} + \delta\, d_{3n+2}\}\}$

$\leq \phi\,\{\max\{d_{3n+1}, d_{3n+2}, (\alpha + \gamma)\, d_{3n+1}, (\beta + \delta)\, d_{3n+2}\}\}$

We have following cases

Case-I: When max = d_{3n+1}, then from the inequality we have, $d_{3n+2} \leq \phi\,(d_{3n+1}) < d_{3n+1}$

Case-II: max = d_{3n+2}, then $d_{3n+2} \leq \phi\,(d_{3n+2}) < d_{3n+2}$, which is a contradiction

Case-III: max = $(\alpha + \gamma)\, d_{3n+1}$ then
$d_{3n+2} \leq \phi\{(\alpha + \delta)\, d_{3n+1}\} < (\alpha + \delta)\, d_{3n+1}$, as $\alpha + \beta + \delta + \gamma < 1/2$ we have, $d_{3n+2} \leq d_{3n+1}$

Case-IV: max = $(\beta + \delta)\, d_{3n+2}$

$d_{3n+2} \le \phi\{(\beta + \delta)\, d_{3n+2}\} < (\beta + \delta)\, d_{3n+2}$. Which is a contradiction as $\alpha + \beta + \gamma + \delta < 1/2$. Hence in either cases $d_{3n+2} \le d_{3n+1}$.From above cases we can say that $d_n \le d_{n-1}$ for every $n \in N$. So, we get $d_n \le q d_{n-1}$ where $q = \alpha + \beta + \gamma + \delta$ i.e. $d_n = G(y_n, y_{n+1}, y_{n+2}) \le q\, G(y_{n-1}, y_n, y_{n+1}) \le q^n G(y_0, y_1, y_2)$.

Also we have $G(x, x, y) \le G(x, y, z)$, hence we get $G(y_n, y_n, y_{n+1}) \le q^n G(y_0, y_1, y_2)$ and

$$G(y_n, y_n, y_m) \le G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \text{----} + \text{-----} + G(y_{m-1}, y_{m-1}, y_m)$$

$$\le q^n G(y_0, y_1, y_2) + q^{n+1} G(y_0, y_1, y_2) + \text{-----} + q^{n-1} G(y_0, y_1, y_2)$$

$$\le \left(\frac{q^n - q^m}{1-q}\right) G(y_0, y_1, y_2) \le \left(\frac{q^n}{1-q}\right) G(y_0, y_1, y_2) \to 0 \text{ as } n \to \infty$$

So, the sequence $\{y_n\}$ is a Cauchy sequence in X and as X is complete $\{y_n\}$ will converge to y in X i.e. $\lim_{n\to\infty} y_n = y$,

$\lim_{n\to\infty} fx_{3n} = \lim_{n\to\infty} gx_{3n+1} = \lim_{n\to\infty} hx_{3n+2} = \lim_{n\to\infty} tx_{3n+1} = \lim_{n\to\infty} sx_{3n+2} = \lim_{n\to\infty} r_{3n+3} = y$. Let h (X) is a closed subset of r (X). Then there exists $u \in X$ such that r u = y Now consider

$$G(fu, gx_{3n+1}, hx_{3n+2}) \le \phi\{\max\{G(gx_{3n+1}, fu, ru), G(hx_{3n+2}, gx_{3n+1}, tx_{3n+1}), G(fu, sx_{3n+2}, hx_{3n+2}),$$
$$\alpha G(fu, ru, gx_{3n+1}) + \gamma G(sx_{3n+2}, fu, ru), \beta G(gx_{3n+1}, tx_{3n+1}, hx_{3n+2}) + \delta G(fu, gx_{3n+1}, hx_{3n+1})\}\}$$

As $n \to \infty$

$$\le \phi\{\max\{G(y, fu, ru), G(y, y, y), G(fu, y, y), \alpha G(fu, ru, y) + \gamma G(y, fu, ru),$$
$$\beta G(y, y, y) + \delta G(fu, y, y)\}\}$$

$$\le \phi\{\max\{G(y, fu, y), G(y, y, y), G(fu, y, y), \alpha G(fu, y, y) + \gamma G(y, fu, y),$$
$$\beta G(y, y, y) + \delta G(fu, y, y)\}\}$$

$$\le \phi\{\max\{G(fu, y, y), (\alpha + \gamma) G(fu, y, y), \delta G(fu, y, y)\}\}$$

We have following cases

Case-I: max = $G(fu, y, y)$ then from above inequality we have.

$G(fu, y, y) \le \phi\{G(fu, y, y)\} < G(fu, y, y)$, which is a contraction.

Case-II: max = $(\alpha + \gamma) G(fu, y, y)$ then from above inequality we have.

$G(fu, y, y) \le \phi\{(\alpha + \gamma) G(fu, y, y)\} < (\alpha + \gamma) G(fu, y, y) \le G(fu, y, y)$. This implies $G(fu, y, y) = 0$, fu = y.

Case-III: max = $\delta G(fu, y, y)$ then from above inequality we have

$G(fu, y, y) \le \phi\{\delta G(fu, y, y)\} < \delta G(fu, y, y) \le G(fu, y, y)$. This implies $G(fu, y, y) = 0$, fu = y. As ru = y we have fu = ru = y. As the pair (f, r) is weakly compatible we have fru = rfu hence fy = ry. Now we prove that fy = y.

$$G(fy, gx_{3n+1}, hx_{3n+2}) \le \phi\{\max\{G(gx_{3n+1}, fy, ry), G(hx_{3n+2}, gx_{3n+1}, tx_{3n+1}), G(fy, sx_{3n+2}, hx_{3n+2}),$$
$$\alpha G(fy, ry, gx_{3n+1}) + \gamma G(sx_{3n+2}, fy, ry), \beta G(gx_{3n+1}, tx_{3n+1}, hx_{3n+2}) + \delta G(fy, gx_{3n+1}, tx_{3n+1})\}\}$$

As $n \to \infty$

$$\le \phi\{\max\{G(y, fy, ry), G(y, y, y), G(fy, y, y), \alpha G(fy, ry, y)$$
$$+ \gamma G(y, fy, ry), \beta G(y, y, y) + \delta G(fy, y, y)\}\}$$

$$\le \phi\{\max\{G(y, fy, fy), G(fy, y, y), \alpha G(fy, fy, y) + \gamma G(fy, fy, y), \delta G(fy, y, y)\}$$

$$\le \phi\{\max\{2G(y, fy, y), G(fy, y, y), (2\alpha + 2\gamma) G(y, fy, y), \delta G(fy, y, y)\}$$

$$\le \phi\{\max\{2G(y, fy, y), (2\alpha + 2\gamma) G(y, fy, y), \delta G(fy, y, y)\}$$

We have following cases

Case-I: max = $2G(y, fy, y)$ then from above inequality we get. $G(y, fy, y) = 0$ i, e fy = y.

Case-II: max = δ G (fy, y, y) then from the equality

$G\ (fy, y, y) \le \phi\{\delta G\ (fy, y, y)\} < \delta G\ (fy, y, y)$, as $\alpha + \beta + \gamma + \delta < 1/2$ so we have

G (fy , y, y) = 0 which implies fy = y.

Case-III: max = $(2\alpha + 2\gamma)$ G (fu, y, y) then

$G\ (fu, y, y) \le \phi\ \{ (2\alpha + 2\gamma)G\ (fy, y, y) \} < (2\alpha + 2\gamma)G\ (fy, y, y) \le G\ (fy, y, y)$ which implies fy = y

As fy = ry = y, we conclude f, r have common fixed point y. As $y = fy \in f(X) \subseteq t(X)$ there exists w such that tw = y. We shall now prove that gw = y.

$G(y, gw, hx_{3n+2}) = G(fy, gw, hx_{3n+2})$

$$\le \phi\ \{\max\{G(gw, fy, ry), G(hx_{3n+2}, gw, tw), G(fy, sx_{3n+2}, hx_{3n+2}), \alpha G(fy, ry, gw) + \gamma G(sx_{3n+2}, fy, ry), \beta G(gw, tw, hx_{3n+2}) + \delta G(fy, gw, tw)\}\}$$

$$\le \phi\ \{\max\{G(gw, y, y), G(y, gw, y), G(y, y, y), \alpha G(y, y, gw) + \gamma G(y, y, y), \beta G(gw, y, y) + \delta (y, gw, y)\}\}$$

$$\le \phi\ \{\max\{G(y, gw, y), \alpha G(y, y, gw), (\beta + \delta) G(gw, y, y)\}$$

We have following cases

Case-I: max = G (y, gw, y) then from the inequality

$G(y, gw, y) \le \phi\{G(y, gw, y)\} < G(y, gw, y)$ which is a contraction.

Case-II: max = α G (y, gw, y) then from the inequality

$G\ (y, gw, y) \le \phi\{\alpha G(y, gw, y)\} < \alpha G\ (y, gw, y)$ which implies G (y, gw, y) = 0 then gw = y. As tw = y = gw and (g, t) being weakly compatible we have gtw = tgw. Then gy = ty. We now prove gy = y.

Consider

$$G(fy, gy, hx_{3n+2}) \le \phi\ \{\max\{G(gy, fy, ry), G(hx_{3n+2}, gy, ty), G(fy, sx_{3n+2}, hx_{3n+2}), \alpha G(fy, ry, gy) + \gamma G(sx_{3n+2}, fy, ry), \beta G(gy, ty, hx_{3n+2}) + \delta G(fy, gy, ty)\}\}$$

$$\le \phi\ \{\max\{G(gy, y, y), G(y, gy, gy), G(y, y, y), \alpha G(y, y, gy) + \gamma G(y, y, y), \beta G(gy, gy, y) + \delta G(y, gy, gy)\}\}$$

$$\le \phi\ \{\max\{G(gy, y, y), 2G(y, gy, y), \alpha G(y, y, gy), (2\beta + 2\delta\ G(y, y, gy)\}$$

$$\le \phi\ \{\max\{2G(y, gy, y), \alpha G(y, y, gy), (2\beta + 2\delta) G(y, y, gy)\}$$

We have following cases

Case-I: max = 2G (y, gy, y) then from the above inequality.

$G(y, gy, y) \le \phi\{2G(y, gy, y)\} < 2G(y, gy, y)$, which implies G (y, gy, y) = 0 then gy = y

Case-II: max = α G (y, y, gy) then from the inequality we have.

$G(y, gy, y) \le \phi\ \{\alpha G(y, y, gy)\} < \alpha G\ (y, y, gy)$.This implies G (y, y, gy) = 0.Thus we have gy = y.

Case-III: max = $(2\beta + 2\delta)$ G (y, y, gy) then from the inequality we have.

$G(y, gy, y) \le \phi\ \{(2\beta + 2\delta) G(y, y, gy)\} < (2\beta + 2\delta) G\ (y, y, gy)$. This implies G (y, y, gy) = 0

So we have gy = y. Thus in either cases gy = y and as gy = ty = y we have y is common fixed point of g, t.

Since $y = gy \in g(X) \subseteq S(X)$ there exist $v \in X$ such that sv = y. We now prove that hv = y.

$G(y, y, hv) = G(fy, gy, hv)$

$$\le \phi\ \{\max\{G(gy, fy, ry), G(hv, gy, ty), G(fy, sv, hv), \alpha G(fy, ry, gy) + \gamma G(sv, fy, ry), \beta G(gy, ty, hv) + \delta G(fy, gy, ty)\}\}$$

$$\le \phi\ \{\max\{G(y, y, y), G(hv, y, y), G(y, y, hv), \alpha G(y, y, y) + \gamma G(y, y, y), \beta G(y, y, hv) + \delta G(y, y, y)\}\}$$

$$\le \phi\ \{\max\{G(hv, y, y), \beta G(y, y, hv)\}$$

We have following cases

Case-I: max = $G(y, y, hv)$ then from the inequality above we have.
$G(y, y, hv) \leq \phi\{G(y, y, hv)\} < G(y, y, hv)$, which implies $G(y, y, hv) = 0$ then $hv = y$

Case-II: max = $\beta G(y, y, hv)$ then from the inequality we have.
$G(y, y, hv) \leq \phi\{\beta G(y, y, hv)\} < \beta G(y, y, hv)$, which implies $hv = y$. Thus in either cases $hv = y$. As $sv = y$ so we have $sv=hv=y$. Since (h, s) are weakly compatible so $hsv = shv$ then $hy = sy$. We now prove that $hy = y$.

Consider

$$G(y, y, hy) = G(fy, gy, hy)$$

$$\leq \phi\{\max\{G(gy, fy, ry), G(hv, gy, ty), G(fy, sy, hy), \alpha G(fy, ry, gy) + \gamma G(sy, fy, ry), \beta G(gy, ty, hy) + \delta G(fy, gy, ty)\}\}$$

$$\leq \phi\{\max\{G(y, y, y), G(hy, y, y), G(y, hy, hy), \alpha G(y, y, y) + \gamma G(hy, y, y), \beta G(y, y, hy) + \delta G(y, y, y)\}\}$$

$$\leq \phi\{\max\{G\ hy, y, y), \gamma G(hy, y, y), \beta G(y, y, hy)$$

We have following cases

Case-I: max = $G(hy, y, y)$ then from the inequality we have.
$G(hy, y, y) \leq \phi\{G(hy, y, y)\} < G(hy, y, y)$ which is a contradiction .

Case-II: max = $\gamma G(hy, y, y)$ then from the inequality we have.
$G(hy, y, y) \leq \phi\{\gamma G(hy, y, y)\} < \gamma G(hy, y, y)$, hence $G(hy, y, y) = 0$ which gives $hy = y$.

Case-III: max = $\beta G(hy, y, y)$ then from the inequality we have. $G(hy, y, y) \leq \phi\{\beta G(hy, y, y)\} < \beta G(hy, y, y)$, which implies $G(hy, y, y) = 0$ which gives $hy = y$.

Thus in either cases $hy = y$. As $sy = hy = y$ therefore y is common fixed point of s and h. Thus y is common fixed point of f, r, s, t, h, g. We shall now prove that the fixed point is unique. Let y' be another fixed point of f, r, g, t, s, h. Then

$$G(y, y, hy') = G(fy, gy, hy')$$

$$\leq \phi\{\max\{G(gy, fy, ry), G(hy', gy, ty), G(fy, hy', sy'), \alpha G(fy, ry, gy) + \gamma G(sy', fy, ry), \beta G(gy, ty, hy') + \delta G(fy, gy, ty)\}\}$$

$$\leq \phi\{\max\{G(y, y, y), G(y', y, y), G(y, y', y'), \alpha G(y, y, y) + \gamma G(y', y, y), \beta G(y, y, y') + \delta G(y, y, y)\}\}$$

$$\leq \phi\{\max\{G(y', y, y), 2G(y, y, y'), \gamma G(y, y', y), \beta G(y, y, y')\}\}$$

$$\leq \phi\{\max\{2G(y, y, y'), \gamma G(y, y, y'), \beta G(y, y, y')\}\}$$

We have following cases

Case-I: max = $\beta G(y, y, y')$ then from the inequality we have.
$G(y, y, y') \leq \phi\{\beta G(y, y, y')\} < \beta G(y, y, y')$, which implies $G(y, y, y') = 0$ then $y = y'$

Case-II: max = $2G(y, y, y')$ then from the inequality we have.
$G(y, y, y') = \phi\{2G(y, y, y')\} < 2G(y, y, y')$, which implies $G(y, y, y') = 0$ as Therefore $y = y'$

Case-III: max = $\gamma G(y, y, y')$ then from the inequality we have.
$G(y, y, y') = \phi\{\gamma G(y, y, y')\} < \gamma G(y, y, y')$, which implies $G(y, y, y') = 0$ as Therefore $y = y'$

Thus the mappings f, r, g, t, h, s have unique common fixed point.

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