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MINIMAL αrω-OPEN SETS AND MAXIMAL αrω-CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, a new class of sets called minimal αrω-open sets and maximal αrω-closed sets in topological spaces are introduced which are subclasses of αrω-open sets and αrω-closed sets respectively. We prove that the complement of minimal αrω-open set is a maximal αrω-closed set and some properties of the new concepts have been studied

Keywords: *Minimal open set, Maximal closed set, Minimal αrω-open set, Maximal αrω-closed set.*

1. INTRODUCTION

In the year 2001 and 2003, F. Nakaoka and N.oda [1] [2] [3] introduced and studied minimal open (resp. minimal closed) sets which are sub classes of open (resp. closed) sets. The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. In the year 2014 R.S Wali & P.S Mandalgeri [4] introduced and studied αrω-closed sets and αrω-open sets in topological spaces.

Definition 1.1: [1] A proper non-empty open subset U of a topological space X is said to be minimal open set if any open set which is contained in U is ϕ or U.

Definition 1.2: [2] A proper non-empty open subset U of a topological space X is said to be maximal open set if any open set which contains U is either X or U.

Definition 1.3: [3] A proper non-empty closed subset F of a topological space X is said to be minimal closed set if any closed set which is contained in F is ϕ or F.

Definition 1.4: [3] A proper non-empty closed subset F of a topological space X is said to be maximal closed set if any closed set which contains F is either X or F.

Definition 1.5: [4] A subset A of (X, τ) is called αrω-closed set if $\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is rω-open in X.

Definition 1.6: [4] A subset A in (X, τ) is called αrω-open set in X if A^c is αrω-closed set in X.

2. MINIMAL αrω-OPEN SETS

Definition 2.1: A proper non-empty αrω-open subset U of X is said to be minimal αrω-open set if any αrω-open set which is contained in U is ϕ or U.

Remark 2.2: Minimal open sets and Minimal αrω-open sets are independent of each other as seen from the following example.

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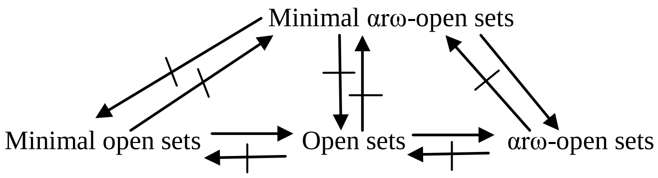
Example 2.3: Let X= {a, b, c, d} be with the topology τ ={X, ϕ, {a}, {c, d}, {a, c, d}}

Minimal open sets are = {{a}, {c, d}}   αrω-open sets are = {X, ϕ, {a}, {b}, {c}, {c, d}, {a, c, d}}

Minimal αrω-open sets are = {{a}, {b}, {c}}

Remark 2.4: From the Known results and by the above example we have the following implications.



Theorem 2.5:
(i) Let U be a minimal αrω-open set and W be a αrω-open set then U∩W= ϕ or U⊂W.
(ii) Let U and V be minimal αrω-open sets then U∩V= ϕ or U=V.

Proof:
(i) Let U be a minimal αrω-open set and W be a αrω-open set. If U∩W= ϕ, then there is nothing to prove but if U∩W≠ ϕ then we have to prove that U⊂W. Suppose U∩W≠ ϕ then U∩W ⊂U and U∩W is αrω-open as the finite intersection of αrω-open sets is a αrω-open set. Since U is a minimal αrω-open set, we have U∩W=U therefore U⊂W.

(ii) Let U and V be minimal αrω-open sets. Suppose U∩V≠ ϕ then we see that U⊂V and V⊂U by (i) Therefore U=V.

Theorem 2.6: Let U be a minimal αrω-open set. If x is an element of U then U⊂W for any open neighbourhood W of x.

Proof: Let U be a minimal αrω-open set and x be an element of U. Suppose there exits an open neighbourhood W of x such that U⊄W then U∩W is a αrω-open set such that U∩W⊂U and U∩W≠ ϕ. Since U is a minimal αrω-open set, we have U∩W=U that is U⊂W. This contradicts our assumption that U⊄W. Therefore U⊂W for any open neighbourhood W of x.

Theorem 2.7: Let U be a minimal αrω-open set, if x is an element of U then U⊂W for any αrω-open set W containing x.

Proof: Let U be a minimal αrω-open set containing an element x. Suppose there exists an αrω- open set W containing x such that U⊄ W then U∩W is an αrω-open set such that U∩W ⊂U and U∩W≠ ϕ Since U is a minimal αrω-open set, we have U∩W=U that is U ⊂W. This contradicts our assumption that U⊄W. Therefore U⊂W for any αrω-open set W containing x.

Theorem 2.8: Let U be a minimal αrω-open set then U= ∩ {W: W is any αrω-open set containing x} for any element x of U

Proof: By theorem 2.7 and from the fact that U is a αrω-open set Containing x, We have U⊂∩ {W: W is any αrω-open set containing x} ⊂U. Therefore we have the result.

Theorem 2.9: Let U be a non-empty αrω-open set then the following three conditions are equivalent.
(i) U is a minimal αrω-open set
(ii) U⊂ αrω-cl(S) for any non-empty subset S of U.
(iii) αrω-cl(U)= αrω-cl(S) for any non-empty subset S of U.

Proof:
(i) => (ii): Let U be a minimal αrω-open set and S be a non-empty subset of U. Let x∈U by theorem 2.7 for any αrω-open set W containing x, S⊂U⊂W which implies S⊂W. Now S=S∩U ⊂S∩W. Since S is non-empty therefore S∩ W≠ ϕ. Since W is any αrω-open set containig x by one of the theorem, we know that, for an x∈X, x∈αrω-cl(A) iff V∩A ≠ ϕ for any every αrω-open set V Containing x, that is x∈U implies x∈αrω-cl(S) which implies U⊂αrω-cl(S) for any non-empty subset S of U.

(ii) => (iii): Let S be a non-empty subset of U, that is $S\subset U$ which implies $\alpha r\omega\text{-cl}(S)\subset\alpha r\omega\text{-cl}(U)$ --(a)
Again from (ii) $U\subset\alpha r\omega\text{-cl}(S)$ for any non-empty subset S of U. Which implies $\alpha r\omega\text{-cl}(U)\subset\alpha r\omega\text{-cl}(\alpha r\omega\text{-cl}(S))=\alpha r\omega\text{-cl}(S)$ i.e., $\alpha r\omega\text{-cl}(U)\subset\alpha r\omega\text{-cl}(S)$ --(b), from (a) and (b), $\alpha r\omega\text{-cl}(U)=\alpha r\omega\text{-cl}(S)$ for any non empty subset S of U.

(iii) => (i): From (iii) we have $\alpha r\omega\text{-cl}(U) = \alpha r\omega\text{-cl}(S)$ for any non-empty subset S of U. Suppose U is not a minimal αrω-open set then there exist a non-empty αrω-open set V such that $V\subset U$ and $V \neq U$. Now there exists an element $a\in U$ such that $a\notin V$ which implies $a\in V^c$ that is $\alpha r\omega\text{-cl}(\{a\})\subset\alpha r\omega\text{-cl}\{(V^c)\} = V^c$, as V^c is a αrω-closed set in X. It follows that $\alpha r\omega\text{-cl}(\{a\})\neq\alpha r\omega\text{-cl}(U)$. This is contradiction to fact that $\alpha r\omega\text{-cl}(\{a\})= \alpha r\omega\text{-cl}(U)$ for any non empty subset {a} of U. Therefore U is a minimal αrω-open set.

Theorem 2.10: Let V be a non-empty finite αrω-open set, then there exists at least one (finite) minimal αrω-open set U such that $U\subset V$.

Proof: Let V be a non-empty finite αrω-open set. If V is a minimal αrω-open set, we may set U=V. If V is not a minimal αrω-open set, then there exists a (finite) αrω-open set V_1 such that $\phi\neq V_1\subset V$. If V_1 is a minimal αrω-open set, we may set $U=V_1$. If V_1 is not a minimal αrω-open set then there exists a (finite) αrω-open set V_2 such that $\phi\neq V_2\subset V_1$. Continuing this process we have a sequence of αrω-open sets $V_k \ldots\subset V_3\subset V_2\subset V_1\subset V$. Since V is a finite set, this process repeats only finitely then finally we get a minimal αrω-open set $U=V_n$ for some positive integer n.

Corollary 2.11: Let X be a locally finite space and V be a non-empty αrω-open set then there exists at least one(finite) minimal αrω-open set such that $U\subset V$.

Proof: Let X be a locally finite space and V be a non empty αrω-open set. Let $x\in V$ since X is a locally finite space we have a finite open set Vx such that $x\in V_x$ then $V\cap V_x$ is a finite αrω-open set. By theorem 2.10 there exist at least one (finite) minimal αrω-open set U such that $U \subset V\cap V_x$ that is $U\subset V\cap V_x\subset V$. Hence there exists at least one (finite) minimal αrω-open set U such that $U \subset V$.

Corollary 2.12: Let V be a finite minimal open set then there exist at least one(finite) minimal αrω-open set U such that $U\subset V$.

Proof: Let V be a finite minimal open set then V is a non-empty finite αrω-open set, by theorem 2.10 there exist at least one (finite)minimal αrω-open set U such that $U\subset V$.

Theorem 2.13: Let U and U_λ be minimal αrω-open sets for any element λ of Λ. If $U \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ then there exists $\lambda\in\Lambda$ an element $\lambda \in \Lambda$ such that $U= U_\lambda$.

Proof: Let $U \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ then $U\cap \bigcup_{\lambda\in\Lambda} U_\lambda = U$ that is $\bigcup_{\lambda\in\Lambda}\left(U\bigcap U_\lambda\right) = U$, also by Theorem 2.5 (ii) $U\bigcap U_\lambda = \phi$ for any $\lambda \in \Lambda$ It follows that there exist an element $\lambda \in \Lambda$ such that $U= U_\lambda$.

Theorem 2.14: Let U and U_λ be minimal αrω-open sets for any element $\lambda\in\Lambda$. If $U = U_\lambda$ for any element λ of Λ then $\bigcup_{\lambda\in\Lambda} U_\lambda \cap U= \phi$

Proof: Suppose that $\bigcup_{\lambda\in\Lambda} U_\lambda \cap U \neq \phi$ that is $\bigcup_{\lambda\in\Lambda}\left(U_\lambda \bigcap U\right) \neq \phi$. then there exists an element $\lambda \in \Lambda$ such that $U \cap U_\lambda \neq \phi$ by theorem 2.5 (ii) we have $U= U_\lambda$, which contradicts the fact that $U\neq U_\lambda$ for any $\lambda \in\Lambda$ then $\bigcup_{\lambda\in\Lambda} U_\lambda \cap U = \phi$.

Theorem 2.15: Let U_λ be a minimal αrω-open set for any element $\lambda \in \Lambda$ and $U_\lambda \neq U_\mu$ for any element λ and μ of Λ with $\lambda\neq \mu$ assume that $|\Lambda| \geq 2$. Let μ be any element of Λ then $\bigcup_{\lambda\in\Lambda-\{M\}} U_\lambda \cap U_\mu = \phi$.

Proof: Put $U= U_\mu$ in theorem 2.14, then we have the result.

Corollary 2.16: Let U_λ be a minimal αrω-open set for any element $\lambda \in \Lambda$ and $U_\lambda \neq U_\mu$ for any element λ and μ of Λ with $\lambda\neq \mu$. If Γ a proper non-empty subset of Λ then $\bigcup_{\lambda\in\Lambda-\Gamma} U_\lambda \cap \bigcup_{\gamma\in\Gamma} U_\gamma = \phi$

Theorem 2.17: Let U_λ and U_γ be minimal αrω-open sets for any element $\lambda \in \Lambda$ and $\gamma \in \Gamma$ If there exists an element γ of Γ such that $U_\lambda\neq U_\gamma$ for any element λ of Λ , then $\bigcup_{\gamma\in\Gamma} U\gamma \not\subset \bigcup_{\lambda\in\Lambda} U_\lambda$

Proof: Suppose that an element γ^1 of Γ satisfies $U_\lambda = U_{\gamma 1}$ for any element λ of Λ. if $\bigcup_{\gamma\in\Gamma} U\gamma \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, then we see $U_{\gamma 1} \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ by theorem 2.13 there exists an element λ of Λ such that $U_{\gamma 1} = U_\lambda$ which is a contradiction. It follows that $\bigcup_{\gamma\in\Gamma} U\gamma \not\subset \bigcup_{\lambda\in\Lambda} U_\lambda$

3. MAXIMAL αrω-closed SETS

Definition 3.1: A proper non-empty αrω-closed subset F of X is said to be Maximal αrω-closed set if any αrω-closed set which contains F is either X or F.

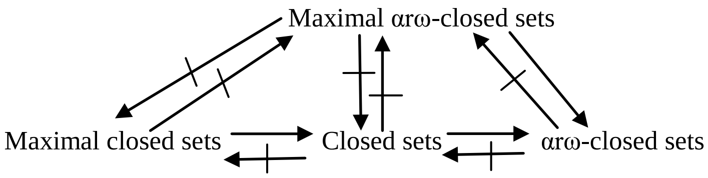
Remark 3.2: Maximal closed sets and Maximal αrω-closed sets are independent each other as seen from the following implication.

Example 3.3: Let X= {a, b, c, d} and τ ={X, ϕ, {a}, {c, d}, {a, c, d}} be a topological space.

Closed sets are ={X, ∅,{b},{a, b},{b, c, d}}Maximal closed sets are = {{a, b},{b, c, d}} αrω-closed sets are ={X, ϕ, {b},{a, b},{b, c, d },{a, c, d},{a, b, d}}

Maximal αrω-closed sets are = {{b, c, d}, {a, c, d}, {a, b, d}}

Remark 3.4: From the known results and by the above example 3.3 we have the following implication.



Theorem 3.5: A proper non-empty subset F of X is Maximal αrω-closed set iff X-F is a minimal αrω-open set.

Proof: Let F be a Maximal αrω-closed set, suppose X-F is not a minimal αrω-open set, then there exists a αrω-open set U≠X−F such that ϕ ≠ U ⊂ X-F that is F⊂X-U and X-U is a αrω-closed set. This contradicts our assumption that F is a minimal αrω- open set.

Conversely, let X-F be a minimal αrω-open set. Suppose F is not a Maximal αrω-closed set then there exist a αrω- closed set E≠ F such that F ⊂E ≠ X that is ϕ ≠ X-E ⊂ X-F and X-E is a αrω-open set. This contradicts our assumption that X-F is a minimal αrω-open set. Therefore F is a Maximal αrω-closed set.

Theorem 3.6:
(i) Let F be a Maximal αrω-closed set and W be a αrω-closed set Then F∪W=X or W⊂F.
(ii) Let F and S be Maximal αrω-closed sets then F∪S=X or F=S

Proof:
(i): Let F be a Maximal αrω-closed set and W be a αrω-closed set if F∪W=X then there is nothing to prove but if F∪W≠ X, then we have to prove that W⊂F. Suppose F∪W≠ X then F ⊂F∪W and F∪W is αrω-closed as the finite union of αrω-closed set is a αrω-closed set we have F∪W=X or F∪W=F. Therefore F∪W=F which implies W⊂F.

(ii): Let F and S be Maximal αrω-closed sets. Suppose F∪S≠ X then we see that F⊂S and S⊂F by (i) therefore F=S.

Theorem 3.7: Let F be a Maximal αrω-closed set. If x is an element of F then for any αrω-closed set S containing x, F∪S=X or S⊂F

Proof: Proof is similar to 2.7 theorems.

Theorem 3.8: Let F_α, F_β, F_γ be Maximal αrω-closed sets such that $F_\alpha \neq F_\beta$ if $F_\alpha \cap F_\beta \subset F_\gamma$, then either $F_\alpha = F_\gamma$ or $F_\beta = F_\gamma$.

Proof: Given that $F_\alpha \cap F_\beta \subset F_\gamma$, if $F_\alpha = F_\gamma$ then there is nothing to prove but if $F_\alpha \neq F_\gamma$ then We have to prove $F_\beta = F_\gamma$.

Now we have $F_\beta \cap F_\gamma = F_\beta \cap (F_\gamma \cap X)$
$= F_\beta \cap (F_\gamma \cap (F_\alpha \cup F_\beta))$ (by theorem 3.6 (ii))
$= F_\beta \cap ((F_\gamma \cap F_\alpha) \cup (F_\gamma \cap F_\beta))$
$= (F_\beta \cap F_\gamma \cap F_\alpha) \cup (F_\beta \cap F_\gamma \cap F_\beta)$
$= (F_\alpha \cap F_\beta) \cup (F_\gamma \cap F_\beta)$ (by $F_\beta \cap F_\gamma \subset F_\beta$)
$= (F_\alpha \cup F_\gamma) \cap F_\beta$
$= X \cap F_\beta$ (since F_α, and F_γ are Maximal αrω-closed sets by thm 3.6 (ii) $F_\alpha \cup F_\gamma = X$)
$= F_\beta$

That is $F_\beta \cap F_\gamma = F_\beta$ implies $F_\beta \subset F_\gamma$, since F_β, F_γ are maximal αrω-closed sets , we have $F_\beta = F_\gamma$.

Theorem 3.9: Let F_α, F_β,F_γ be Maximal αrω-closed sets which are different from each other then $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\gamma)$.

Proof: Let $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\gamma)$ which implies $(F_\alpha \cap F_\beta) \cup (F\gamma \cap F_\beta) \subset (F_\alpha \cap F_\gamma) \cup (F\gamma \cap F_\beta)$ which implies $(F_\alpha \cup F_\gamma) \cap F_\beta \subset F_\gamma \cap (F_\alpha \cup F_\beta)$ since by theorem 3.6 (ii) $F_\alpha \cap F_\gamma = X$ and $F_\alpha \cap F_\beta = X$ which implies $X \cap F_\beta \subset F_\gamma \cap X$ which implies $F_\beta \subset F_\gamma$. From the definition of Maximal αrω-closed set it follows that $F_\beta = F_\gamma$.

This is contradiction to the fact that Let F_α, F_β,F_γ are different from each other. Therefore $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\gamma)$.

Theorem 3.10: Let F be a Maximal αrω-closed set and x be an element of F then $F \neq \cup${S: S is a αrω-closed set containing x such that $F \cup S \neq X$}.

Proof: By theorem 3.7 and from the fact that F is a αrω-closed set containing x we have $F \subset \cup${S: S is a αrω-closed set containing x such that $F \cup S \neq X$}$\subset F$ therefore we have the result.

Theorem 3.11: Let F be a Proper non-empty co-finite αrω-closed subset then there exists (co-finite) Maximal αrω-closed set E such that $F \subset E$.

Proof: Let F be a non-empty co-finite αrω-closed set. If F is a Maximal αrω-closed set, we may set E=F. If F is not a Maximal αrω-closed set, then there exists a (co-finite) αrω-closed set F_1 such that $F \subset F_1 \neq X$. If F_1 is a Maximal αrω-closed set, we may set $E = F_1$. If F_1 is not a Maximal αrω-closed set ,then there exists a (co-finite) αrω-closed set sets F_2 such that $F \subset F_1 \subset F_2 \neq X$ continuing this process we have a sequence of αrω-closed sets $F \subset F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots$
since F is a co-finite set, this process repeats only finitely then finally we get a minimal αrω-open set $E=F_n$ for some positive integer n.

Theorem 3.12: Let F be a Maximal αrω-closed set. If x is an element of X-F then $X-F \subset E$ for any αrω-closed set containing set E containing x

Proof: Let F be a Maximal αrω-closed set and $x \in X-F$. $E \not\subset F$ for any αrω-closed set E containing x then $E \cup F = X$ by theorem 3.6(ii). Therefore $X-F \subset E$

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REFERENCES

1. F.Nakaoka and F.Oda, Some Application of minimal open sets, Int. J. Math. Math. sci. Vol 27, No.8, 471-476(2001).
2. F.Nakaoka and F.Oda, Some Properties of Maximal open sets, Int. J. Math. Math. sci. Vol 21, No.21, 1331-1340(2001).
3. F.Nakaoka and F.Oda, on minimal closed sets, Proceeding of topological spaces and It's application, 5, 19-21(2003).
4. R.S.wali and P.S.Mandalgri, On α Regular ω-closed sets in topological spaces, Int. J. of Math. Archive 5(10), 2014, 68-76.

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