International Research Journal of Pure Algebra-4(12), 2014, 648-652
Available online through www.rjpa.info ISSN 2248–9037

ON THE H-GROUP

Swapnil Srivastava[1], Ajit Paul[2] and Punish Kumar[3,*]

[1,2,3] Department of Mathematics & Statistics, SHIATS, Allahabad, (U.P.), India.

(Received On: 18-12-14; Revised & Accepted On: 25-12-14)

ABSTRACT

In this paper, with the help of $\tilde{p}$ -map, we have defined an H-transversal for an H-group and then we have shown that $\tilde{p}(G)$ is an H-group.

AMS Subject Classification: *20J99.*

Key Words: *H-space, H-group, $\tilde{p}$ -map, H-transversal.*

1. INTRODUCTION

We have observed that Ramji Lal and Ungar & Foguel in their papers [3, 4] have studied transversals in groups in abstract sense. In our paper namely, H-transversal in H-groups [5], we have studied transversals in topological sense and then we have shown that there is a canonical H-group structure on $\tilde{p}(G)$ with respect to which the inclusion $\tilde{p}(G) \xrightarrow{i} G$ is an H-subgroup of an H-group (G, μ) where map $\tilde{p}$ be an H-transversal.

In this paper, using $\tilde{p}$ -map, we have defined another H-transversal for an H-group. Then we have proved that $\tilde{p}(G)$ is also an H-group.

Note: Throughout the paper $\approx$ represents homotopy between two maps.

1. $\tilde{p}$ -map and H-Space

In the present section, we have defined $\tilde{p}$ -map, topological group, H-space, etc [1, 2, 6].

Definition 2.1: Let G be a group with identity e. A map $\tilde{p}$ from G to G satisfying the following properties:

(i) $\tilde{p}(e) = e$

(ii) $\tilde{p}^2 = \tilde{p}$

(iii) $\tilde{p}(g_1g_2) = \tilde{p}(\tilde{p}(g_1)g_2)$, is called a $\tilde{p}$ - map.

Example 2.2: Identity map I on the group G is a $\tilde{p}$ - map.

Proposition 2.3: Let G be a group with identity e. Let H be a subgroup of G and S be a right transversal (with identity) to H in G .Since each $g \in G$ can be uniquely written as hx where $h \in H$ and $x \in S$. Then a map $\tilde{p}: G \to G$ defined by $\tilde{p}(g) = x$ is a $\tilde{p}$ - map.

Proof: Proof follows from proposition 2.3 of [6].

Corresponding author: Punish Kumar[3,*]
[1,2,3] Department of Mathematics & Statistics, SHIATS, Allahabad, (U.P.), India.

Proposition 2.4: Let G be a group with identity e and $\tilde{p}: G \to G$ be a $\tilde{p}$-map. Then the set $H = \{g \in G : \tilde{p}(g) = e\}$ is a subgroup of G.

Proof: Proof follows from proposition 2.6 of [6].

Proposition 2.5: Let G be a group with identity e and $\tilde{p}: G \to G$ be a $\tilde{p}$-map. Then the subset $S = \{p(g) : g \in G\}$ of G is a right transversal with identity to the subgroup $H = \{g \in G : \tilde{p}(g) = e\}$ in G.

Proof: Proof follows from proposition 2.7 of [6].

Definition 2.6: A **topological group** G is a group that is also a topological space, satisfying the requirements that the map of $G \times G$ into G sending $x \times y$ into $x \cdot y$, and the map of G into G sending x into x^{-1}, are continuous.

Definition 2.7: A nonempty topological space with a base point is called a pointed topological space.

Definition 2.8: A pointed topological space G with base point e_0 together with a continuous multipication $\mu : G \times G \to G$ for which the unique constant map $c : G \to G$ defined by $c(x) = e_0$, is a homotopy identity, that is, each composite $G \xrightarrow{(c\times 1)} G \times G \xrightarrow{\mu} G$ and $G \xrightarrow{(1\times c)} G \times G \xrightarrow{\mu} G$ is homotopic to identity map $(1_G : G \to G)$, is called an **H-space**.

Example 2.8: Any topological group is an H-space.

3. H-GROUP AND H-TRANSVERSAL

In this section, by defining H-group and H-transversal, we have shown that $\tilde{p}(G)$ is an H-group. (Theorem 3.13)

Definition 3.1: Let G be an H-space. The continuous multipication $\mu : G \times G \to G$ is said to be homotopy **associative** if $\mu \circ (\mu \times 1) \approx \mu \circ (1 \times \mu)$.

Definition 3.2: Let G be an H-space. A continuous function $\varphi : G \to G$ is called a **homotopy inverse** for G and μ if each of the composites $G \xrightarrow{(\varphi\times 1)} G \times G \xrightarrow{\mu} G$ and $G \xrightarrow{(1\times\varphi)} G \times G \xrightarrow{\mu} G$ is homotopic to homotopic identity $c : G \to G$.

Definition 3.3: A homotopy associative H-space with a homotopy inverse satisfies the group axioms upto homotopy. Such a pointed space is called an **H- group**.

Example 3.4: Any topological group is also an H-group.

Definition 3.5: The continuous multiplication $\mu : G \times G \to G$ in an H-group G is said to be homotopy abelian if $\mu \circ T \approx \mu$ where the map $T : G \times G \to G \times G$ is defined by $T(p_1, p_2) = (p_2, p_1)$.

Definition 3.6: An H-group with homotopy abelian multiplication is called an **abelian H-group**.

Definition 3.7: If G and G' are H-groups with multiplication μ and μ' respectively. A continuous map $\alpha : G \to G'$ is called a **homomorphism** if $\alpha \circ \mu \approx \mu' \circ (\alpha, \alpha)$.

Definition 3.8: If G and G' are H-groups with multiplication μ and μ' respectively. A homomorphism $\alpha : G \to G'$ is called an **H-map** if $\alpha \circ c \approx c' \circ \alpha$ where c and c' are homotopy identity for G and G' respectively.

Definition 3.9: An equivalence class of monomorphism in the category of H-groups is called an **H-subgroup**. More explicitly, let (G, μ) be an H-group. An H-subgroup is an H-group (K, ν) together with an H-map $\varphi : K \to G$ if given any H-group (L, η) and two H-maps $f_1, f_2 : L \to K$ such that $\varphi \circ f_1 \approx \varphi \circ f_2 \Rightarrow f_1 \approx f_2$. Thus $[\varphi]$ is a class of monomorphisms in the category of H-groups (objects are H-groups and morphisms are equivalence class of H-maps). This described a subgroup as an equivalence class of H-maps.

Definition 3.10: Consider the set $S=\{\varphi : K \to G \text{ is an } H-map : (K,\nu) \text{ is an } H-subgroup \text{ of an H-group } (G,\mu)\}$. Define two H-maps $\varphi_1 : K_1 \to G$ and $\varphi_2 : K_2 \to G$ **equivalent** if there exists H-maps $h_1 : K_1 \to K_2$ and $h_2 : K_2 \to K_1$ such that $h_2 \circ h_1 \approx I_{K_1}$, $h_1 \circ h_2 \approx I_{K_2}$, $\varphi_2 \circ h_1 \approx \varphi_1$ and $\varphi_1 \circ h_2 \approx \varphi_2$.

Proposition 3.11: Let (X, x_0) and (Y, y_0) be two pointed topological spaces. Then $\Omega X = \{\omega : \omega : I \to X \text{ is a loop based at } x_0\}$ is an H-group with continuous multiplication μ. Similarly $\Omega Y = \{\omega : \omega : I \to Y \text{ is a loop based at } y_0\}$ is an H-group with continuous multiplication ν. Let $f : (Y, y_0) \to (X, x_0)$ is a continuous map. Then $(\Omega Y, \nu)$ is an H-subgroup together with an H-map $(\Omega Y, \nu) \xrightarrow{\Omega f} (\Omega X, \mu)$.

***Proof*:** Proof follows from proposition 2.14 of [5].

Definition 3.12: An **H-transversal** in an H-group (G,μ) is a continuous identity preserving map $\tilde{p} : G \to G$ such that

(i) $\tilde{p}^2 \approx \tilde{p}$

(ii) $\tilde{p} \circ \mu \approx \tilde{p} \circ \mu \circ (\tilde{p} \times 1_G)$

Theorem 3.13: Let (G,μ) be an H-group with base point identity element e of the group G. Let $\tilde{p}$ be an H-transversal in an H-group (G,μ). Then $\tilde{p}(G)$ is an H-group with respect to the operation ν defined as follows $\nu(\tilde{p}(g_1), \tilde{p}(g_2)) = (\tilde{p} \circ \mu)(\tilde{p}(g_1), \tilde{p}(g_2))$ for all $g_1, g_2 \in G$.

Proof: Since $\tilde{p}$ be an H-transversal in an H-group (G,μ) then we have $\tilde{p} \circ \mu \approx \tilde{p} \circ \mu \circ (\tilde{p} \times 1_G)$

Thus there is a homotopy $H : G \times G \times I \to G$ such that

$H((g_1, g_2), 0) = \tilde{p}(\mu(g_1, g_2))$

$H((g_1, g_2), 1) = \tilde{p}(\mu(\tilde{p}(g_1), g_2))$ for all $g_1, g_2 \in G$

Define a product $\nu : \tilde{p}(G) \times \tilde{p}(G) \to \tilde{p}(G)$ by

$$\begin{aligned}
\nu(\tilde{p}(g_1), \tilde{p}(g_2)) &= H((\tilde{p}(g_1), \tilde{p}(g_2)), 0) \\
&= \tilde{p}(\mu(\tilde{p}(\tilde{p}(g_1)), \tilde{p}(g_2))) \\
&= \tilde{p}(\mu(\tilde{p}(g_1), \tilde{p}(g_2))) \\
&= (\tilde{p} \circ \mu)(\tilde{p}(g_1), \tilde{p}(g_2))
\end{aligned}$$

We show that $(\tilde{p}(G), \nu)$ is an H-group.

Since $\tilde{p}$ and μ are continuous so is ν.

Now,

(i) Since G is an H-group so the constant map $c_G : G \to G$ given by $c_G(g) = e$ is a homotopy identity that is, $\mu \circ (c_G \times 1_G)$ is homotopic to identity map 1_G and similarly $\mu \circ (1_G \times c_G)$ is also homotopic to identity map 1_G.

Now for $\tilde{p}(g) \in \tilde{p}(G)$, we have

$$(\mu \circ (c_G \times 1_G))\,\tilde{p}(g) = \mu \circ (c_G(\tilde{p}(g)), \tilde{p}(g)) = \mu(e, \tilde{p}(g))$$

Let $c_{\tilde{p}(G)} : \tilde{p}(G) \to \tilde{p}(G)$ denote constant map on $\tilde{p}(G)$ defined by $c_{\tilde{p}(G)}(\tilde{p}(g)) = \tilde{p}(e) = e$. Replacing G above by $\tilde{p}(G)$.

We have

$$
\begin{aligned}
(\nu\circ(c_{\tilde{p}(G)}\times 1_{\tilde{p}(G)}))(\tilde{p}(g)) &= (\nu(c_{\tilde{p}(G)}(\tilde{p}(g)),\tilde{p}(g))))\\
&= \nu(e,\tilde{p}(g))\\
&= (\tilde{p}\circ\mu)(e,\tilde{p}(g))\\
&\approx (\tilde{p}\circ\mu)((c_G\times 1_G)(\tilde{p}(g)))\\
&\approx \tilde{p}((\mu\circ(c_G\times 1_G))(\tilde{p}(g)))\\
&\approx \tilde{p}(1_G(\tilde{p}(g)))\\
&\approx \tilde{p}(\tilde{p}(g))\\
&\approx \tilde{p}(g)\\
&\approx 1_{\tilde{p}(G)}(\tilde{p}(g))
\end{aligned}
$$

Thus $\nu\circ(c_{\tilde{p}(G)}\times 1_{\tilde{p}(G)})\approx 1_{\tilde{p}(G)}$

Similarly, $\nu\circ(1_{\tilde{p}(G)}\times c_{\tilde{p}(G)})\approx 1_{\tilde{p}(G)}$

Thus $c_{\tilde{p}(G)}$ is homotopic identity for $(\tilde{p}(G),\nu)$.

(ii) Let $\varphi: G\to G$ be homotopy inverse for (G,μ). So $\mu\circ(\varphi\times 1_G)$ and $\mu\circ(1_G\times\varphi)$ are homotopic to homotopy identity c_G for G.

Now,

$$
\begin{aligned}
(\nu\circ(1_{\tilde{p}(G)}\times\varphi_{\tilde{p}(G)}))(\tilde{p}(g)) &= \nu(\tilde{p}(g),\varphi_{\tilde{p}(G)}(\tilde{p}(g)))\\
&= \nu(\tilde{p}(g),\tilde{p}(g_1)) \quad \text{for some } g_1\in G\\
&= (\tilde{p}\circ\mu)(1_G(\tilde{p}(g)),\varphi(\tilde{p}(g))) \qquad [\because \tilde{p}(g)\in\tilde{p}(G)\subset G]\\
&= (\tilde{p}\circ\mu)((1_G\times\varphi)(\tilde{p}(g)))\\
&= \tilde{p}((\mu\circ(1_G\times\varphi))(\tilde{p}(g)))\\
&\approx \tilde{p}(c_G(\tilde{p}(g)))\\
&\approx \tilde{p}(e)\\
&\approx e\\
&\approx c_{\tilde{p}(G)}(\tilde{p}(g))
\end{aligned}
$$

Thus, $\nu\circ(1_{\tilde{p}(G)}\times\varphi_{\tilde{p}(G)})\approx c_{\tilde{p}(G)}$

Similarly, we can show that $\nu\circ(\varphi_{\tilde{p}(G)}\times 1_{\tilde{p}(G)})\approx c_{\tilde{p}(G)}$

Hence $\varphi_{\tilde{p}(G)}$ is homotopy inverse for $\tilde{p}(G)$.

(iii) Since (G,μ) is associative. So we have $\mu\circ(\mu\times 1_G)\approx\mu\circ(1_G\times\mu)$.

Now, replacing G by $\tilde{p}(G)$, we have to show that $\nu\circ(1_{\tilde{p}(G)}\times\nu)\approx\nu\circ(\nu\times 1_{\tilde{p}(G)})$

Now,

$$
\begin{aligned}
(\nu\circ(1_{\tilde{p}(G)}\times\nu))(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3)) &= \nu(\tilde{p}(g_1),\nu(\tilde{p}(g_2),\tilde{p}(g_3)))\\
&= \nu(\tilde{p}(g_1),(\tilde{p}\circ\mu)(\tilde{p}(g_2),\tilde{p}(g_3)))\\
&\approx (\tilde{p}\circ\mu)(\tilde{p}(\tilde{p}(g_1)),\tilde{p}(\mu(\tilde{p}(g_2),\tilde{p}(g_3)))) \qquad [\text{Since } \tilde{p}^2\approx\tilde{p}\,]\\
&\approx ((\tilde{p}\circ\mu)\circ\tilde{p}\times\tilde{p})(\tilde{p}(g_1),\mu(\tilde{p}(g_2),\tilde{p}(g_3)))
\end{aligned}
$$

$$\approx ((\tilde{p}\circ\mu\circ(\tilde{p}\times 1_G))\circ\tilde{p}\times\tilde{p})((1_G\times\mu)(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3)))$$

[Since $\tilde{p}\circ\mu\approx\tilde{p}\circ\mu\circ(\tilde{p}\times 1_G)$]

$$\approx ((\tilde{p}\circ\mu)\circ((\tilde{p}\times 1_G)\circ\tilde{p}\times\tilde{p}))((1_G\times\mu)(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3)))$$

$$\approx ((\tilde{p}\circ\mu)\circ(\tilde{p}\times 1_G))((1_G\times\mu)(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3)))$$

[Since $(\tilde{p}\times 1_G)\circ\tilde{p}\times\tilde{p})=\tilde{p}\times 1_G$ on $\tilde{p}(G)$]

$$\approx (\tilde{p}\circ\mu)((1_G\times\mu)(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3)))$$

$$\approx (\tilde{p}\circ(\mu\circ(1_G\times\mu)))(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3))$$

$$\approx (\tilde{p}\circ(\mu\circ(\mu\times 1_G)))(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3))$$

$$\approx (\tilde{p}\circ\mu)((\mu\times 1_G)(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3)))$$

$$\approx (\tilde{p}\circ\mu\circ(\tilde{p}\times 1_G))(\mu(\tilde{p}(g_1),\tilde{p}(g_2)),\tilde{p}(g_3))$$

$$\approx (\tilde{p}\circ\mu\circ((\tilde{p}\times 1_G)\circ(\tilde{p}\times\tilde{p}))(\mu(\tilde{p}(g_1),\tilde{p}(g_2)),\tilde{p}(g_3))$$

$$\approx (\tilde{p}\circ\mu\circ(\tilde{p}\times 1_G))(\tilde{p}(\mu(\tilde{p}(g_1),\tilde{p}(g_2))),\tilde{p}(\tilde{p}(g_3)))$$

$$\approx (\tilde{p}\circ\mu)((\tilde{p}\circ\mu)(\tilde{p}(g_1),\tilde{p}(g_2)),\tilde{p}(g_3))$$

$$\approx \nu(\nu(\tilde{p}(g_1),\tilde{p}(g_2)),\tilde{p}(g_3))$$

$$\approx \nu((\nu\circ 1_{\tilde{p}(G)})(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3))$$

$$\approx (\nu\circ(\nu\circ 1_{\tilde{p}(G)}))(\tilde{p}(g_1),\tilde{p}(g_2),\tilde{p}(g_3))$$

Thus $(\nu\circ(1_{\tilde{p}(G)}\times\nu))\approx(\nu\circ(\nu\circ 1_{\tilde{p}(G)}))$

Hence $(\tilde{p}(G),\nu)$ is an H-group.

4. CONCLUSION

In our paper, extension of groups using $\tilde{p}$-maps [6], we have shown that G be an extension of the subgroup $H=\{g\in G:\tilde{p}(g)=e\}$ with a right transversal $S=\{p(g):g\in G\}$. In this paper, we have tried to find out its approach in topological sense by making $\tilde{p}$ to be continuous map. We are hopeful that using category theory, one can find some relationship between algebraic and topological approach.

REFERENCES

[1] E.H. Spanier, Algebraic Topology, McGraw Hill Book Company (1966).
[2] J.R.Munkres, Toplogy: A First Course, Prentice Hall (1975).
[3] T. Foguel, A. A. Ungar. Involutory decomposition of Groups into twisted subgroups and subgroups, J. Group Theory, 3(2000), 27-46.
[4] R. Lal. Transversals in Groups, J. Algebra, 181(1996), 70-81.
[5] S. Srivastava, P. Kumar, H-transversals in H-groups, International J. of Algebra, HIKARI, Vol. 8 (2014), No. 15, 705-712.
[6] S. Srivastava, P. Kumar, Extension of Groups using p-tilda Maps, (communicated with the Journal Communications in Algebra).

Source of Support: Nil, Conflict of interest: None Declared