International Research Journal of Pure Algebra-4(12), 2014, 653-657
Available online through www.rjpa.info ISSN 2248–9037

# A STUDY ON THE TAYLOR_CESÀRO PRODUCT SUMMABILITY METHOD OF FOURIER SERIES

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***(Received On: 20-10-14; Revised & Accepted On: 31-10-14)***

***ABSTRACT***

*In the present paper, we will study on the ($T_nC_2$) product summability method of Fourier series under the general condition. In this paper we will prove a new theorem on the degree of approximation of function belonging to lip $\alpha$ class by ($T_n$ $C_2$) means of its Fourier series.*

***Keywords:*** *Degrees of approximation, Taylor_Cesàro mean, Fourier series.*

## 1. INTRODUCTION

Let f be $2\pi$ - periodic and integrable in the Lebesgue sense. The Fourier series associated with f at a point x is given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + b_n \sin nx$$

A function $f \in lip\alpha$, if

$$f(x \pm t) - f(x) = 0\left(\left|t^{\alpha}\right|\right) \; for \; 0 < \alpha \leq 1$$

**Definition 1.1:** The degree of approximation of a function f: R → R by a trigonometric polynomial $T_n$ of degree *n* is given by

$$\| T_n - f \|_{\infty} = \sup\left\{\left|T_n(x) - f(x)\right| : x \in p\right\}$$

**Definition 1.2:** Let $\sum u_n$ be a given infinite series with sequence of its $n^{th}$ partial sum $\{S_n\}$. The (C,2) transform is defined as the nth partial sum of (C,2) summability and is given by

$$\sigma_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1)\, S_k \to S \quad as \; n \to \infty$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C,2), method.

**Definition 1.3:** A given sequence $\{S_n\}$ is said to be Taylor summable, if

$$(T_n) = \sum_{k=0}^{n} u_{n,k} \; S_k \to S \; as \; n \to \infty,$$

then the (c, 2) transform of Taylor means defines the ($T_n$ $C_2$) transform of the partial sums $\{S_n\}$ of the series (1.1).

Thus, if $(T_nC_2) = \sum_{k=0}^{n} u_{n,n-k}, \sigma_{n-k} \to S \; as \; n \to \infty$

*then* $\sum_{n=0}^{\infty} u_n$ *is said to be* $T_nC_2$ summable to $S$

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**Remark 1.1:** We shall use following notations:

$$(i)\ \varnothing(t) = f(x+t) - f(x-t) - 2f(x)$$

$$(ii)\ D(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{u_{n,n-k}}{(n-k+2)}\frac{\sin^2(n-k+2)t/2}{\sin^2 t/2}$$

## 2. MAIN THEOREM

The degree of approximation of functions belonging to Lip α class by various summability methods of the Fourier series of f have been studied by several researchers like Alexits [1], Chandra [2], Holland [3] and Qureshi [4] etc. Here in the present paper, we obtain the degree of approximation of function f ∈ Lip α, class by Taylor_Cesàro product simmability method we prove the following :

**Theorem 2.1:** If f: R→ R is $2\pi$ periodic and lebesgue integrable on $[-\pi\ \pi]$ and f ∈ Lip$\alpha$, then the degree of approximation of function by Taylor_Cesàro product means of the Fourier series, satisfies for n=0, 1,2 ....,

$$\| T_n C_2(x) - f(x) \|_\infty = \begin{cases} 0\left(\dfrac{1}{(n+2)^\alpha}\right); 0<\alpha<1 \\ 0\left(\dfrac{\log(n+2)\pi e}{n+2}\right); \alpha = 1 \end{cases}$$

where $T_n = a_{n,k}$ is a non- negative, monotonic and non-increasing sequence of real constant such that

$$\left|\sum_{k=0}^{n} u_{n,\,n-k}\right| = 0(1). \tag{2.1}$$

for the proof of the theorem, the following lemmas are required :

**Lemma 2.1:** For $O \le t \le \frac{1}{n+2}; D(n,t) = O(n+2)$

**Proof:** we have

$$|D(n,t)| = \left|\frac{1}{2\pi}\sum_{k=0}^{n}\frac{u_{n,n-k}}{(n-k+2)}\frac{\sin^2(n-k+2)t/2}{\sin^2 t/2}\right|$$

$$\le \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{u_{n,\,n-k}}{(n-k+2)}\frac{(n-k+2)^2(n-k+2)t^2/\pi^2}{t^2/\pi^2}\right|$$

$$= O(n+2)\left|\sum_{k=0}^{n} u_{n,\,n-k}\right|$$

= O (n+2) by (2.1)

**Lemma 2.2:** For $1/(n+2) \le t \le \pi; D(n,t) = 0\left(\frac{1}{(n+2)t^2}\right)$

**Proof:** We have

$$|D(n,t)| = \left|\frac{1}{2\pi}\sum_{k=o}^{n}\frac{u_{n,\,n-k}}{n-k+2)}\frac{\sin^2(n-n+2)t/2}{\sin^2 t/2}\right|$$

Using Jordan's lemma $\sin\frac{t}{2} \geq, \frac{t}{\pi} and \sin kt \leq 1;$ we have

$$\leq \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{u_{n,\,n-k}}{(n-k+2)}\frac{1}{t^2/\pi^2}\right|$$

$$= 0\left(\frac{1}{n+2}\right)\left|\sum_{k=0}^{n}u_{n,\,n-k}\right|$$

$$= 0\left(\frac{1}{(n+2)t^2}\right), by (2.1)$$

**3. PROOF OF THE THEOREM**

Let $S_n(x)$ denote the nth partial sum of the series (1.1) at t = *x*, then the following Titchmarch [5], we have

$$\sigma_n(x) - f(x) = \frac{2(n-k+1)}{2\pi(n+1)(n+2)}\int_0^{\pi}\frac{\sin^2(n+2)t/2}{\sin^2 t/2}dt$$

Now, the Taylor, transform of the sequence $\{\sigma_n\}$ is given by

$$\sum_{k=0}^{n}u_{n,\,n-k}\{\sigma_n(x) - f(x)\} = \frac{2}{2\pi}\int_0^{\pi}\varnothing(t)\sum_{k=0}^{n}\frac{u_{n,\,n-k}}{(n-k+2)}\frac{\sin^2(n-k+2)t/2}{\sin^2 t/2}dt; at\ k = 0$$

Or

$$T_nC_2(x) - f(x) = 2\int_0^{\pi}\varnothing(t)D(n,t)dt$$

$$= 2[I_1 + I_2] \text{ Say} \tag{3.1}$$

Let us consider $I_1$ first

$$|I_1| = \left|\int_0^{1/n+2}\varnothing(t)D(n,t)dt\right|$$

$$\leq \int_0^{1/n+2}|\varnothing(t)||D(n,t)|dt$$

$$= \int_0^{1/n+2}\varnothing(t^\alpha)O(n+2)dt, \text{ by lemma 2.1 and } \varnothing(t) \in Lip\ \alpha$$

$$= 0(n+2)\int_0^{\frac{1}{n+2}}t^\alpha dt$$

$$= 0\left(\frac{1}{(n+2)^\alpha}\right);\ 0 < \alpha \leq 1 \tag{3.2}$$

Finally, we consider $I_2$.

$$|I_2| = \left|\int_{1/n+2}^{\pi}\varnothing(t)D(n,t)dt\right|$$

$$\leq \int_{1/n+2}^{\pi}|\varnothing(t)||D(n,t)|dtby\ kmma2.2and\ \varnothing(t) \in Lip\ \alpha$$

$$= 0\left(\frac{1}{n+2}\right)\int_{1/n+2}^{\pi}t^{\alpha-2}dt$$

 

$$=\begin{cases} 0\left(\frac{1}{(n+2)}\right)\left(\frac{t^{\alpha-1}}{\alpha-1}\right)_{\frac{1}{n+2}}^{\pi} & :0<\alpha<1 \\ 0\left(\frac{1}{n+2}\right)(logt)_{\frac{1}{n+2}}^{\pi} :\alpha=1 \end{cases}$$

$$=\begin{cases} 0\left(\frac{1}{(n+2)}\right)\left[\frac{1}{\alpha-1}\left(\frac{1}{(n+2)^{\alpha-1}}-\frac{1}{(\pi)^{1-\alpha}}\right)\right];0<\alpha<1 \\ 0\left(\frac{1}{(n+2)}\right)\left[log\pi+\log(n+2)\right] \quad ;\alpha=1 \end{cases}$$

$$=\begin{cases} 0\left\{\left(\frac{1}{(n+2)^{\alpha}}\right) \quad ;0<\alpha<1 \right. \\ \quad 0\left(\frac{\log(n+2)\pi}{(n+2)}\right);\alpha=1 \end{cases} \tag{3.3}$$

Now combining (3.1), (3.2) and (3.3); we get

$$\left|T_nC_2(x)-f(x)\right|=\begin{cases} 0\left(\frac{1}{(n+2)^{\alpha}}\right);0<\alpha<1 \\ 0\left(\frac{1}{(n+2)}\right)+0\left(\frac{\log(n+2)\pi}{(n+2)}\right);\alpha=1 \end{cases}$$

$$\left|T_nC_2(x)-f(x)\right|=\begin{cases} 0\frac{1}{(n+2)^{\alpha}};0<\alpha<1 \\ 0\left(\frac{\log(n+2)\pi e}{(n+2)}\right);\alpha=1 \end{cases}$$

Thus,

$$\| T_nC_2(x)-f(x)\|_{\infty} \quad \sup_{-\pi\le x\le\pi}\left|T_nC_2(x)-f(x)\right|$$

$$T_nC_2(x)-f(x)_{\infty}=\begin{cases} 0\frac{1}{(n+2)^{\alpha}};0<\alpha<1 \\ 0\left(\frac{\log(n+2)\pi e}{(n+2)}\right);\alpha=1 \end{cases}$$

This completes the proof of the theorem.

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**Source of Support: Nil, Conflict of interest: None Declared**