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EXTENSION OF MATRIX PROPERTIES TO FULL TRANSFORMATION SEMIGROUP

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ABSTRACT

Some matrix properties were extended to full transformation semigroup to determine linear dependence and independence of the elements.

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1. INTRODUCTION

Let X_n be the set of the first n natural numbers as X_n = {1, 2, 3. . . n} and let T_n denote the full transformation semigroup of X_n. The matrix representation of transformation semigroup, S is defined in [2] as follows:

For $\alpha \in S$, let $\Psi(\alpha) = (m_{i,j})_{i,j=1}$ denote the n x n matrix such that $m_{i,j} = \begin{cases} 1, \alpha(j)=i \\ 0, otherwise. \end{cases}$. In this work, for each $\alpha \in T_n$, m (α) indicates matrix representation of the corresponding α .

For example, let X_n = {1, 2, 3} and $\alpha, \beta \in T_n$ where $\alpha : 1 \to 1,\ 2 \to 3,\ 3 \to 1$ is represented by the matrix $m(\alpha) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\beta : 1 \to 3,\ 2 \to 1,\ 3 \to 1$ is represented by the matrix $m(\beta) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. It is worth noting here that the composition of mapping $\alpha\beta \equiv m(\beta)m(\alpha)$. Let $S_n \subseteq T_n$ be the group of permutations in T_n and let $Sing_n = T_n - S_n$ be the subsemigroup of T_n consisting of the singular transformations. The semigroup $Sing_n$ is idempotent - generated and its idempotent rank is $\frac{n(n-1)}{2}$ as studied in [3] and [4].

An inverse semigroup S is defined if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$.

The matrix representation of semigroups defines algebra of semigroups over a set of natural numbers, N as a vector space over N. The operation of multiplication is defined satisfying for every $\alpha_1, \alpha_2, \alpha_3 \in T_n$ and every $n \in N$:

(i) m[$\alpha_1(\alpha_2 + \alpha_3)] = m[\alpha_1\alpha_2] + m[\alpha_1\alpha_3]$,

(ii) m[$(\alpha_2 + \alpha_3)\alpha_1] = m[\alpha_2\alpha_1] + m[\alpha_3\alpha_1]$,

(iii) m[$n(\alpha_2\alpha_1)] = m[(n\alpha_2)\alpha_1] = m[\alpha_{2(}(n\alpha_1)]$.

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Since the semigroup is associative, then the algebra defined on T_n is associative as
(iv) $(\alpha_1\alpha_2)\alpha_3 = \alpha_1(\alpha_2\alpha_3)$.

2. THE DETERMINANT OF T_n

The determinant of T_n, ΔT_n was obtained and used in determining the linear dependence and independence of T_n. Some of the results obtained are outlined in this section.

Lemma 2.1: *The determinant of $T_n - S_n$ is zero.*

Proof: Singular transformations leave gaps in between points in X_n, which pave way for zero rows.

Theorem 2.1: $\Delta T_n \in [-1, 1]$.

Proof: The only entries in the matrix representation of T_n are 0's and 1's, which make the determinant zero for any $\alpha \in T_n$ having at least a zero row. This follows from lemma 2.1. The permutation group S_n has determinant 1 so long there is no fix point and $i, j \in X_n$ such that $i \to j$ and $j \to i$ only once, otherwise the determinant is -1. Thus the determinant of each element $\alpha \in T_n$ was obtained having the range $-1 \leq \Delta T_n \leq 1$.

Lemma 2.2: *If α is an idempotent element then $\alpha^2 \equiv m(\alpha^2) \Rightarrow \alpha \equiv m(\alpha)$.*

This lemma simply shows that matrix representation preserves idempotency. It should also be noted that identity element in T_n is equivalent to its corresponding matrix identity.

Theorem 2.2: *Let $m(\alpha_1)$ be any element in the matrix representation of the semigroup T_n. Then the following are equivalent:*

(i) The determinant of $m(\alpha_1)$ is not zero,
(ii) $m(\alpha_1)$ is non – singular, i.e. the rank $\alpha_1 = n$,
(iii) $m(\alpha_1)$ is invertible, i.e. $m(\alpha_1)$ has an inverse $[m(\alpha_1)]^{-1}$.

Proof: $(i) \Rightarrow ((ii)$: It is known that the determinant of a matrix is zero if an entire row is zero or two rows (or columns) are equal or a row (or a column) is a constant multiple of another row (or column). If any of these three is not visible for $m(\alpha_1)$, then the determinant of $m(\alpha_1) \neq 0$. This implies that $\alpha_1 \in S_n \subset T_n$. Hence rank $\alpha_1 = n$.

$(ii) \Rightarrow ((iii)$: The rank of α_1 depends on its length of image. If the length of image of α_1 is n, then α_1 is of rank n. Since $\alpha_1 \in S_n$ then rank of α_1 = n and there exist an element $\alpha_1^{-1} \in S$ such that $\alpha_1 = \alpha_1\alpha_1^{-1}\alpha_1$ and $\alpha_1^{-1} = \alpha_1^{-1}$ α_1 α_1^{-1}. The equivalent matrix representation of α_1 and α_1^{-1} is true. Thus $m(\alpha_1)$ is invertible.

$(iii) \Rightarrow ((i)$: If the inverse of a matrix exist, it means that it is not a singular transformation. The existence of the inverse of α_1 indicates that it is of rank n and the determinant is not zero because there is no zero row (or column) in $m(\alpha_1)$.

3. THE SYMMETRIZATION OF T_n

A matrix M is symmetric if $M=M^T$ and only square matrices can be symmetric. The matrix $M=\left[m_{ij}\right]_1^n$ is combinatorially symmetric as defined in [5] if $m_{ij} \neq 0$ implies $m_{ji} \neq 0$. A theorem of Frobenius [1] states that every finite dimensional square matrix over an arbitrary field can be expressed as the product of two symmetric matrices, one of which can be chosen non-singular. This theorem is true for matrix representation of full transformation semigroup. The symmetric elements in T_n are combinatorially symmetric.

Theorem 3.1: *If $\alpha \in T_n$ and $\Delta(\alpha)$ is negative then $m(\alpha)$ is symmetric.*

Proof: The determinant of m(α) is known to be negative by inspection if there are points $i, j \in X_n$ that interchange once as $i \to j$ and $j \to i$ while other elements are fixed. The interchange in only two points implies that m (α) is symmetric.

For example, the mapping $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ has a negative determinant by inspection since $1 \to 3,\ 3 \to 1$ and 2 & 4 are fixed.

It can be verified that every symmetric matrix has negative determinant.

Theorem 3.2: *Let m (α) be an n x n symmetric matrix and let Δ_r be the upper left r x r submatrix for all $1 \le r \le n$. Let $^{-}\Delta_r$ denote negative determinant. The upper left r x r determinant of a symmetric matrix alternate sign and hence the matrix is negative definite.*

Proof: Theorem3.1 showed that symmetric matrix implies negative determinant. Assuming that $\Delta_1 \ge 0, \Delta_2 \ge 0 \ldots \Delta_{r-1} \ge 0$, but $\Delta_r < 0$ then it shows that at least one of the upper left r x r determinants of a symmetric matrix is negative.

The corresponding $m(\alpha_1)$ of the example above is given as $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The upper left r x r determinants are

Δ_1 = 0, Δ_2 = 0, Δ_3 = -1 and Δ_4 = $\Delta(\alpha_1)$ = -1.

4. LINEAR DEPENDENCE AND INDEPENDENCE OF FULL TRANSFORMATION SEMIGROUP

Matrix representation of transformation semigroup linearizes the semigroup. For any $\alpha \in T_n$, α is linearly dependent if the determinant, ΔT_n = 0 and linearly independent if $\Delta T_n \neq 0$.

Dentition: Let $\alpha \in T_n$. A linear transformation is a function α : $X_n \to X_n$ with the matrix representation denoted by m(α) and the following properties :

1. For any $\alpha_1, \alpha_2 \in T_n$, then $m(\alpha_1 + \alpha_2)$= m(α_1) + m(α_2),
2. For any m(α)$\in \Psi(\alpha)$,r $\in$ R then m(α r) = rm(α).

Lemma 4.1: Identity element is the only linearly independent element in the set of idempotents E(T_n), of T_n using the corresponding matrix representation.

Proof: The proof follows from the fact that the determinant of singular transformation is zero (Lemma 2.1 and Theorem2.2) and the identity map is not singular.

Theorem 4.1: The cardinality of linearly independent elements, $|LIT_n|$ of T_n is n!.

Proof: The symmetric group $S_n \subseteq T_n$, is linearly independent since the determinant is not zero.

Theorem 4.2: The cardinality of linearly dependent elements, $|LDT_n|$ of T_n is n^n – n!

Proof: The remaining elements in T_n that are linearly dependent are written as $T_n - S_n$. The result follows from Lemma 2.1.

5. CONCLUSION

The symmetric elements in T_n are combinatorially symmetric and the determinant of T_n, $\Delta T_n \in [-1, 1]$.

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