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# PRIME TERNARY SUBSEMIMODULES IN TERNARY SEMIMODULES

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***ABSTRACT***

*Let $M$ be a ternary semimodule over a ternary semiring $R$. We introduce the notion of prime ternary subsemimodule of $M$ which is a generalization of a prime subsemimodule introduced by Atani [1] and hence we extend the results of semimodules, ternary semirings and partial semimodules to ternary semimodules over ternary semirings. Prime ternary subsemimodules of a multiplication ternary semimodule over ternary semiring are characterized. Also we prove prime avoidance theorem for ternary semimodules over ternary semirings.*

***Mathematics Subject Classification:*** *16Y60, 16Y99.*

***Keywords:*** *Ternary semimodule; ternary semiring; prime ternary subsemimodule; subtractive ternary subsemimodule, multiplication ternary semimodule.*

## 1. INTRODUCTION

Dutta and Kar [8], introduced the notion of ternary semimodule over ternary semiring. Characterizations of the partitioning and subtractive ternary subsemimodules of ternary semimodules and direct sum of partitioning ternary subsemimodules of ternary semimodules are obtained by Chaudhari and Bendale [3]. Prime ideals of ternary semirings is studied by Dutta and Kar [7]. In [1], Atani, has extended this work for semimodules over semirings. Prime avoidance theorem for ideals in ternary semirings is given by Chaudhari and Ingale [4]. In this present paper, we introduce the concept of prime ternary subsemimodule and hence we extend some basic results of ternary semirings [7], semimodules over semirings [1] and partial semimodules [10] to ternary semimodules over ternary semirings. Also we prove the prime avoidance theorem for ternary semimodules over ternary semirings.

For the definitions of monoid and semiring we refer [9] and for ternary semiring we refer [6]. All ternary semirings in this paper are commutative with nonzero identity. Denote the set of all non-positive, positive, and non-negative integers respectively by $\mathbb{Z}_0^-$, $\mathbb{N}$ and $\mathbb{Z}_0^+$. The set $\mathbb{Z}_0^-$ is a ternary semiring under usual addition and ternary multiplication of non-positive integers. An ideal $I$ of a ternary semiring $R$ is called a subtractive ideal (= $k$-ideal) if $a, a+b \in I, b \in R$, then $b \in I$. A proper ideal $P$ of a ternary semiring $R$ is called

1) prime if whenever $IJK \subseteq P$ with $I, J, K$ are ideals of $R$, then either $I \subseteq P$ or $J \subseteq P$ or $K \subseteq P$;
2) completely prime if whenever $abc \in P$ where $a, b, c \in R$, then either $a \in P$ or $b \in P$ or $c \in P$.

Since $R$ is commutative ternary semiring, both the concepts prime ideal and completely prime ideal are coincide.

Let $R$ be a ternary semiring. A left ternary $R$-semimodule is a commutative monoid $(M, +)$ with additive identity $0_M$ for which we have a function $R \times R \times M \to M$, defined by $(r_1, r_2, x) \longmapsto r_1 r_2 x$ called ternary scalar multiplication, which satisfies the following conditions for all elements $r_1, r_2, r_3$ and $r_4$ of $R$ and all elements $x$ and $y$ of $M$:

1) $(r_1 r_2 r_3) r_4 x = r_1 (r_2 r_3 r_4) x = r_1 r_2 (r_3 r_4 x)$;
2) $r_1 r_2 (x + y) = r_1 r_2 x + r_1 r_2 y$;
3) $r_1 (r_2 + r_3) x = r_1 r_2 x + r_1 r_3 x$;
4) $(r_1 + r_2) r_3 x = r_1 r_3 x + r_2 r_3 x$;
5) $1_R 1_R x = x$;
6) $r_1 r_2 0_M = 0_M = 0_R r_2 x = r_1 0_R x$.

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Throughout this paper, by a ternary $R$-semimodule we mean a left ternary semimodule over a ternary semiring $R$.

Clearly, every ternary semiring $R$ is a temary $R$-semimodule. Also every ternary semiring $R$ is ternary $(\mathbb{Z}_0^-,+,\cdot)$-semimodule [3]. A non-empty subset $N$ of a left ternary $R$-semimodule $M$ is called ternary subsemimodule of $M$ if $N$ is closed under addition and closed under ternary scalar multiplication.

If $\{N_i : i \in \mathbb{N}\}$ is a family of ternary subsemimodules of a ternary $R$-subsemimodule $M$, then
 i) $\bigcap_{i\in\mathbb{N}} N_i$ is a ternary subsemimodule of $M$ and it is the largest ternary subsemimodule of $M$ contained in each $N_i$.
 ii) $\sum_{i\in\mathbb{N}} N_i = \{\sum_i x_i : x_i \in N_i\}$ is the smallest ternary subsemimodule of $M$ containing each $N_i$.

Union of ternary subsemimodules of $M$ need not be a ternary subsemimodule.

A ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ is called subtractive ternary subsemimodule (= ternary $k$-subsemimodule) if $x.x+y \in N, y \in M$, then $y \in N$.

If $N$ is a proper ternary subsemimodule of a ternary $R$-semimodule $M$ and $A$ is a non-empty subset of $M$, then we denote
 i) $(N : m) = \{r \in R : rsm \in N \; for \; all \; s \in R\}$ where $m \in M$.
 ii) $(N : A) = \{r \in R : rsA \subseteq N \; for \; all \; s \in R\}$.

Clearly, $(N : m)$ is an ideal of $R$ and $(N : A) = \cap \{(N : m) : m \in A\}$. Since intersection of arbitrary family of ideals is again an ideal, $(N : A)$ is an ideal of $R$ and it will be called as associated ideal of $N$ with respect to $A$.

**Theorem 1.1:** Let $N$ be a subtractive ternary subsemimodule of a ternary $R$-semimodule $M$ and $A$ be a non-empty subset of $M$. Then $(N : A)$ is a subtractive ideal of $R$.

**Proof:** Easy.

**Definition 1.2:** Let $A$ be a non-empty subset of a ternary $R$-semimodule $M$. Then the ternary subsemimodule generated by $A$ is the intersection of all ternary subsemimodules of $M$ containing $A$ and it is denoted by $RRA$ i.e. $RRA = \langle A \rangle = \cap \{N : N \; is \; a \; ternary \; subsemimodule \; of \; M \; with \; A \subseteq N\}$.

**Theorem 1.3:** Let $M$ be a ternary $R$-semimodule. Then for any non-empty subset $A$ of $M, RRA = \{\sum_{finite} rsa : r,s \in$ *R, a∈A.*

**Proof:** Clearly, $\{\sum_{finite} rsa : r,s \in R, a \in A\}$ is a ternary subsemimodule of $M$ containing $A$. Hence, $RRA \subseteq \{\sum_{finite} rsa : r,s \in R, a \in A\}$. Other inclusion is trivial.

## 2. PRIME TERNARY SUBSEMIMODULES

In this section, we introduce the notion of prime ternary subsemimodule of ternary semimodule over ternary semiring which is a generalization of prime subsemimodule introduced by Atani [1]. Moreover we extend the results of semimodules, ternary semirings and partial semimodules to ternary semimodules over ternary semirings. We begin with key definition of this paper:

**Definition 2.1:** A proper ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ is called prime ternary subsemimodule if $r,s \in R, n \in M$ and $rsn \in N$ then $r \in (N : M)$ or $s \in (N : M)$ or $n \in N$.

The following lemma is easy to prove and it will be used in the subsequent theory.

**Lemma 2.2:** Let $N$ be a proper ternary subsemimodule of a ternary $R$-semimodule $M$. Then the following statements are equivalent:
 i) $N$ is a prime ternary subsemimodule;
 ii) If whenever $IJD \subseteq N$ where $I, J$ are ideals of $R$ and $D$ is a ternary subsemimodule of $M$, then $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$.

 

**Theorem 2.3:** If $K$ is a prime ternary subsemimodule of a ternary $R$-semimodule $M$, then $(K : M)$ is a prime ideal of $R$.

**Proof:** Suppose that $K$ is a prime ternary subsemimodule of $M$. Let $a, b, c \in R, abc \in (K : M)$ and $a \notin (K : M), b \notin (K : M)$. Now $ab(crM) = (abc)rM \subseteq K$ for all $r \in R$. Since $K$ is a prime ternary subsemimodule, $crM \subseteq K$ for all $r \in R$. Hence $c \in (K : M)$. Thus, $(K : M)$ is a prime ideal of $R$.

The following example shows that the converse of the Theorem 2.3 is not true.

**Example 2.4:** Consider the ternary $\mathbb{Z}_0^-$ -semimodule $\mathbb{Z}_0^- \times \mathbb{Z}_0^- (= M)$ under the ternary scalar multiplication $* : (r, s, (m, n)) \mapsto (rsm, rsn)$. Clearly, $K = \{0\} \times (-8)\mathbb{Z}_0^-\mathbb{Z}_0^-$ is a ternary subsemimodule of $M$ but it is not a prime ternary subsemimodule of $M$ because $(-2) * (-2) * (0, -2) = \big((-2)(-2)0, (-2)(-2)(-2)\big) = (0, -8) \in K$ but $-2 \notin (K : M)$ and $(0, -2) \notin K$. Clearly, $(K : M) = \{0\}$ is a prime ideal of $\mathbb{Z}_0^-$.

**Definition 2.5:** Let $M$ be a ternary $R$-semimodule. Then $M$ is said to be multiplication ternary semimodule if for each ternary subsemimodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IRM$.

**Lemma 2.6:** If $M$ is a multiplication ternary $R$-semimodule and $N$ is a ternary subsemimodule of $M$, then $N = (N: M)RM$.

**Proof:** Since $M$ is a multiplication ternary semimodule, there exists an ideal $I$ of $R$ such that $N = IRM$. Then $IRM \subseteq N \Rightarrow I \subseteq (N: M)$. So $N = IRM \subseteq (N: M)RM \subseteq N$. Hence $N = (N: M)RM$.

Now we show that the converse of the Theorem 2.3 is true for the multiplication ternary semimodules.

**Theorem 2.7:** Let $N$ be a ternary subsemimodule of a multiplication ternary $R$-semimodule $M$. Then $N$ is a prime ternary subsemimodule of $M$ if and only if $(N: M)$ is a prime ideal of $R$.

**Proof:** Proof of the direct part follows from Theorem 2.3. Conversely, suppose that $(N: M)$ is a prime ideal of $R$. Let $I, J$ be ideals of $R$ and $K$ be a ternary subsemimodule of $M$ such that $IJK \subseteq N$. Since $M$ is a multiplication ternary semimodule, there exists an ideal $L$ of $R$ such that $K = LRM$. Now $N \supseteq IJK = IJ(LRM) = (IJL)RM$. Hence $IJL \subseteq (N: M)$. Since $(N: M)$ is prime ideal of $R, I \subseteq (N: M)$ or $J \subseteq (N: M)$ or $L \subseteq (N: M)$. Thus $I \subseteq (N: M)$ or $J \subseteq (N: M)$ or $K = LRM \subseteq N$. Hence by Lemma 2.2, $N$ is a prime ternary subsemimodule of $M$.

**Theorem 2.8:** A ternary $R$-semimodule $M$ is a multiplication ternary semimodule if and only if for each $m \in M$, there exists an ideal $I$ of $R$ such that $RRm = IRM$.

**Proof:** Suppose $M$ is a multiplication ternary $R$-semimodule. Let $m \in M$. Then $RRm$ is a ternary subsemimodule of $M$. Hence there exists an ideal $I$ of $R$ such that $RRm = IRM$. Conversely, suppose that for each $m \in M$, there exists an ideal $I$ of $R$ such that $RRm = IRM$. Let $N$ be a ternary subsemimodule of $M$. Then for $n \in N$, there exists an ideal $I_n$ of $R$ such that $RRn = I_n R\,M$. Denote $I = \sum_{n \in N} I_n$. Then $I$ is an ideal of $R$. Now $N = \sum_{n \in N} RRn = \sum_{n \in N} I_n RM = (\sum_{n \in N} I_n)RM = IRM$. Hence $M$ is a multiplication ternary $R$-semimodule.

**Definition 2.9:** Let $N_1, N_2, N_3$ be ternary subsemimodules of a multiplication ternary $R$-semimodule $M$ such that $N_1 = IRM, N_2 = JRM$ and $N_3 = KRM$ for some ideals $I, J, K$ of $R$. Then the ternary multiplication of $N_1, N_2$ and $N_3$ is defined as $N_1N_2N_3 = (IRM)(JRM)(KRM) = (IJK)RM$.

**Definition 2.10:** Let $M$ be a multiplication ternary $R$-semimodule and $m_1, m_2, m_3 \in M$ such that $RRm_1 = I_1RM, RRm_2 = I_2RM$ and $RRm_3 = I_3RM$ for some ideals $I_1, I_2$ and $I_3$ of $R$. Then the multiplication of $m_1, m_2$ and $m_3$ is defined as $m_1m_2m_3 = (I_1RM)(I_2RM)(I_3RM) = (I_1I_2I_3)RM$.

Now the following theorem gives a characterization of prime ternary subsemimodule of a multiplication ternary $R$-semimodule $M$.

**Theorem 2.11:** Let $N$ be a proper ternary subsemimodule of a multiplication ternary $R$-semimodule $M$. Then the following statements are equivalent :

1) $N$ is a prime ternary subsemimodule;
2) For any ternary subsemimodules $U, V$ and $W$ of $M, UVW \subseteq N$ implies $U \subseteq N$ or $V \subseteq N$ or $W \subseteq N$;
3) For any $m_1, m_2, m_3 \in M, m_1m_2m_3 \subseteq N$ implies $m_1 \in N$ or $m_2 \in N$ or $m_3 \in N$.

 

**Proof: (1)⇒(2):** Suppose that $N$ is a prime ternary subsemimodule of $M$ and let $U, V, W$ be ternary subsemimodules of $M$ such that $UVW \subseteq N$. Since $M$ is a multiplication ternary semimodule, there exist ideals $I, J$ and $K$ of $R$ such that $U = IRM, V = JRM$ and $W = KRM$. Now $(IJK)RM = UVW \subseteq N \Rightarrow IJK \subseteq (N:M)$. By Theorem 2.3, $(N:M)$ is a prime ideal of $R$. So $I \subseteq (N:M)$ or $J \subseteq (N:M)$ or $K \subseteq (N:M)$. Hence $U = IRM \subseteq N$ or $V = JRM \subseteq N$ or $W = KRM \subseteq N$.

**(2) ⇒(3):** Suppose for any ternary subsemimodules $U, V$ and $W$ of $M, UVW \subseteq N$ implies $U \subseteq N$ or $V \subseteq N$ or $W \subseteq N$. Let $m_1, m_2, m_3 \in M$ such that $m_1 m_2 m_3 \subseteq N$. Since $M$ is a multiplication ternary semimodule, there exist ideals $I, J$ and $K$ of $R$ such that $RRm_1 = IRM, RRm_2 = JRM$ and $RRm_3 = KRM$. Now $(RRm_1)(RRm_2)(RRm_3) = m_1 m_2 m_3 \subseteq N \Rightarrow RRm_1 \subseteq N$ or $RRm_2 \subseteq N$ or $RRm_3 \subseteq N$. Hence $m_1 \in N$ or $m_2 \in N$ or $m_3 \in N$.

**(3) ⇒(1):** Suppose for any $m_1, m_2, m_3 \in M, m_1 m_2 m_3 \subseteq N$ implies $m_1 \in N$ or $m_2 \in N$ or $m_3 \in N$. We prove $(N:M)$ is a prime ideal of $R$. Let $I, J$ and $K$ be ideals of $R$ such that $IJK \subseteq (N:M)$. Then $(IJK)RM \subseteq N$. Suppose that $I \nsubseteq (N:M)$, $J \nsubseteq (N:M)$ and $K \nsubseteq (N:M)$. Therefore $IRM \nsubseteq N$, $JRM \nsubseteq N$ and $KRM \nsubseteq N$. Choose $i \in I,\ j \in J,\ k \in K,\ r_1,\ r_2,\ r_3 \in R$ and $m_1, m_2, m_3 \in M$ such that $ir_1m_1 \in IRM\backslash N,\ jr_2m_2 \in JRM\backslash N$ and $kr_3m_3 \in KRM\backslash N\ldots(*)$. Now $(ir_1m_1)(jr_2m_2)(kr_3m_3) \subseteq (IRM)(JRM)(KRM) = (IJK)RM \subseteq N \Rightarrow ir_1m_1 \in N$ or $jr_2m_2 \in N$ or $kr_3m_3 \in N$, a contradiction to $(*)$. Hence $(N:M)$ is a prime ideal of $R$. Therefore by Theorem 2.7, N is a prime ternary subsemimodule of $M$.

**Theorem 2.12:** Every prime ternary subsemimodule $N$ of a multiplication ternary $R$-semimodule $M$ contains a minimal prime ternary subsemimodule.

**Proof:** Take $\mathcal{A} = \{H : H\ is\ a\ prime\ ternary\ subsemimodule\ of\ M, H \subseteq N\}$. Since $N \in \mathcal{A}$, $(\mathcal{A}, \subseteq)$ is a non-empty partially ordered set. Let $\{H_i : i \in \mathbb{N}\}$ be a descending chain of ternary subsemimodules of $M$ such that $H_i \subseteq N$ for all $i \in \mathbb{N}$ and let $H' = \bigcap_{i \in \mathbb{N}} H_i$. Then $H'$ is a ternary subsemimodule of $M$ and $H' \subseteq N$. Now we prove $H'$ is prime ternary subsemimodule of $M$. Let $m_1, m_2, m_3 \in M$ such that $m_1 m_2 m_3 \subseteq H'$ and $m_1 \notin H', m_2 \notin H'$. Then $m_1 \notin H_k$ for some $k \in \mathbb{N}$ and $m_2 \notin H_l$ for some $l \in \mathbb{N}$. Take $n = \max\{k, l\} \Rightarrow m_1, m_2 \notin H_n \Rightarrow m_3 \in H_n$, since $m_1 m_2 m_3 \subseteq H' \subseteq H_n$ and $H_n$ is a prime ternary subsemimodule and by Theorem 2.11. For any $i \leq n,\ H_i \supseteq H_n$ and hence $m_3 \in H_i$...(1). For any $i > n, H_i \subseteq H_n \Rightarrow m_1, m_2 \notin H_i$, and hence $m_3 \in H_i$ for all $i > n$...(2). From (1) and (2), we get $m_3 \in H'$. Hence $H' \in \mathcal{A}$. Then by Zorn's Lemma, $\mathcal{A}$ has a minimal element. Hence the theorem.

## 3. PRIME AVOIDANCE THEOREM

In this section, we prove prime avoidance theorem for ternary semimodules over ternary semirings.

**Definition 3.1:** The $k$-closure of a ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ is defined by $\overline{N} = \{x \in M : x + a_1 = a_2\ for\ some\ a_1,\ a_2 \in N\}$.

**Proposition 3.2:** Let $N, K$ be ternary subsemimodules of a ternary $R$-semimodule $M$. Then

1) $\overline{N}$ is a ternary subsemimodule of $M$ containing $N$;
2) $N$ is a ternary $k$-subsemimodule of $M \Leftrightarrow \overline{N} = N$;
3) $\overline{\overline{N}} = \overline{N}$;
4) $N \subseteq K \Rightarrow \overline{N} \subseteq \overline{K}$;
5) $\overline{N \cap K} \subseteq \overline{N} \cap \overline{K}$.

**Proof:** Easy.

Consider the ternary semiring $R = (\mathbb{Z}^- \cup \{-\infty\}, max, \cdot)$[5]. Then $M = (\mathbb{Z}^- \cup \{-\infty\}, max)$ is a ternary $R$-semimodule. For $n \in \mathbb{Z}^-$, we denote $T_n = \{r \in \mathbb{Z}^- : r \leq n\} \cup \{-\infty\}$, is an ideal of $R$ [5]. Hence $T_n$ is a ternary subsemimodule of $M$. The following example shows that equality in Proposition 3.2 (5) may not hold.

**Example 3.3:** Consider the ternary semiring $R = (\mathbb{Z}^- \cup \{-\infty\}, max, \cdot)$ and the ternary subsemimodules $N = \{-4\} \cup T_{-8}, K = \{-6, -7\} \cup T_{-10}$ of the ternary $R$-semimodule $M = (\mathbb{Z}^- \cup \{-\infty\}, max)$. Then $N \cap K = T_{-10}$ is a subtractive ternary subsemimodule of $M$ and hence by Proposition 3.2(2), $\overline{N \cap K} = T_{-10}$. But $\overline{N} = T_{-4}$ and $\overline{K} = T_{-6}$. Hence $\overline{N} \cap \overline{K} = T_{-6}$. Now $\overline{N \cap K} \subsetneq \overline{N} \cap \overline{K}$.

**Theorem 3.4:** Let $L$ be a ternary subsemimodule of a ternary $R$-semimodule $M$. If $L_1, L_2$ are ternary $k$-subsemimodules of $M$ such that $L \subseteq L_1 \cup L_2$, then $L \subseteq L_1$ or $L \subseteq L_2$.

 

**Proof:** Suppose that $L \subseteq L_1 \cup L_2, L \nsubseteq L_1$ and $L \nsubseteq L_2$. Choose $l_1 \in L \backslash L_1$ and $l_2 \in L \backslash L_2$. Since $L \subseteq L_1 \cup L_2, l_1 \in L_2$ and $l_2 \in L_1$. Now $l_1 + l_2 \in L_1$ or $l_1 + l_2 \in L_2$. Since $L_1, L_2$ are ternary $k$-subsemimodules, either $l_1 \in L_1$ or $l_2 \in L_2$, a contradiction. Hence $L \subseteq L_1$ or $L \subseteq L_2$.

**Definition 3.5:** Let $L, L_1, L_2, L_3, \ldots, L_n$ be ternary subsemimodules of ternary $R$-semimodule $M$ and $L \subseteq L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$. Then $L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$ is said to be efficient covering of $L$ if $L \nsubseteq \bigcup_{\substack{i=1, \\ i \neq j}}^{n} L_i$ for any $j \in \{1, 2, 3, \ldots, n\}$.

**Definition 3.6:** Let $L, L_1, L_2, L_3, \ldots, L_n$ be ternary subsemimodules of ternary $R$-semimodule $M$ and $L = L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$. Then $L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$ is said to be efficient union of $L$ if $L \neq \bigcup_{\substack{i=1, \\ i \neq j}}^{n} L_i$ for any $j \in \{1, 2, 3, \ldots, n\}$.

**Lemma 3.7:** Let $L$ be a ternary subsemimodule of a ternary $R$-semimodule $M$ and $L_1, L_2, L_3, \ldots, L_n$ be ternary $k$-subsemimodules of $M$. If $L = L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$ is an efficient union, then $\bigcap_{i=1}^{n} L_i = \bigcap_{\substack{i=1 \\ i \neq j}}^{n} L_i$ for all $1 \leq j \leq n$.

**Proof:** Let $1 \leq j \leq n$ and let $x \in \bigcap_{\substack{i=1 \\ i \neq j}}^{n} L_i$. Since $L = L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$ is an efficient union, $L \nsubseteq \bigcup_{\substack{i=1, \\ i \neq j}}^{n} L_i$. So there exists $y \in L$ such that $y \notin \bigcup_{\substack{i=1, \\ i \neq j}}^{n} L_i$ ...(1). Hence $y \in L_j$. Now $x + y \in L = L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$. Suppose that $x + y \in L_i$ for some $i \neq j$. Since $x \in L_i$ and $L_j$ is a ternary $k$-subsemimodule, $y \in L_i$, a contradiction to (1). Hence $x + y \in L_j$. Since $y \in L_j$ and $L_j$ is a ternary $k$-subsemimodule, $x \in L_j$. Hence $x \in \bigcap_{i=1}^{n} L_i$. Now $\bigcap_{\substack{i=1 \\ i \neq j}}^{n} L_i \subseteq \bigcap_{i=1}^{n} L_i$. Other inclusion is trivial.

The following theorem is essential to prove for prime avoidance theorem.

**Theorem 3.8:** Let $L \subseteq L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$ be an efficient covering consisting of ternary $k$-subsemimodules of a ternary $R$-semimodule $M$ where $n > 2$. If $(L_j : M) \nsubseteq (L_k : M)$ for any $k \neq j$, then $L_k$ is not a prime ternary subsemimodule of $M$.

**Proof:** Since $L \subseteq L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$ is an efficient covering, $L = (L \cap L_1) \cup (L \cap L_2) \cup (L \cap L_3) \cup \ldots \cup (L \cap L_n)$ is an efficient union. Then by Lemma 3.7, $\bigcap_{\substack{j=1 \\ j \neq k}}^{n} (L \cap L_j) = \bigcap_{j=1}^{n} (L \cap L_j) \subseteq L \cap L_k$ ...(1). Since $L \nsubseteq L_k$, there exists $l_k \in L \backslash L_k$. Suppose $L_k$ is a prime ternary subsemimodule of $M$. Then $(L_k : M)$ is a prime ideal of $R$. Since $(L_j : M) \nsubseteq (L_k : M)$, there exists $s_j \in (L_j : M) \backslash (L_k : M)$. Then $s = s_1^{n+1} s_2 \ldots s_{k-1} s_{k+1} \ldots s_n \in (L_j : M) \backslash (L_k : M)$. Hence $srl_k \in L \cap L_j$ for all $j \neq k$, for all $r \in R$ and $sr'l_k \notin L \cap L_k$ for some $r' \in R$. So $\bigcap_{\substack{j=1 \\ j \neq k}}^{n} (L \cap L_j) \nsubseteq L \cap L_k$, a contradiction to (1). Thus, $L_k$ is not a prime ternary subsemimodule.

Now we prove prime avoidance theorem for ternary semimodules over ternary semirings.

**Theorem 3.9:** (***Prime Avoidance Theorem***) Let $L$ be a ternary subsemimodule of a ternary $R$-semimodule $M$ and $L_1, L_2, L_3, \ldots, L_n$ be ternary $k$-subsemimodules of $M$ such that atmost two of $L_i$'s are not prime and for any $j \neq k$, $(L_j : M) \nsubseteq (L_k : M)$. If $L \subseteq L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_n$, then $L \subseteq L_k$ for some $k$.

**Proof:** Let $L \subseteq L_{i_1} \cup L_{i_2} \cup L_{i_3} \cup \ldots \cup L_{i_m}$ be the efficient covering where $i_1, i_2, i_3, \ldots, i_m \in \{1, 2, 3, \ldots, n\}$ and $1 \leq m \leq n$. Suppose $m = 2$. Then $L \subseteq L_{i_1} \cup L_{i_2}$ is an efficient covering...(1). By Theorem 3.4, $L \subseteq L_{i_1}$ or $L \subseteq L_{i_2}$, a contradiction to (1). Suppose that $m \geq 3$. Then by assumption, there exists at least one prime ternary subsemimodule $L_{i_j}$ for some $i_j$ which is impossible by Theorem 3.8. Hence $m = 1$. Now $L \subseteq L_k$ for some $k \in \{1, 2, 3, \ldots, n\}$.

**Theorem 3.10:** Let $P$ be a prime ternary subsemimodule and $N_1, N_2, N_3, \ldots, N_n$ be ternary $k$-subsemimodules of a multiplication ternary $R$-semimodule $M$. Then $\bigcap_{i=1}^{n} N_i \subseteq P$ if and only if $N_i \subseteq P$ for some $j \in \{1, 2, 3, \ldots, n\}$.

**Proof:** Suppose that $\bigcap_{i=1}^{n} N_i \subseteq P$. Then $(\bigcap_{i=1}^{n} N_i : M) \subseteq (P : M) \Rightarrow \bigcap_{i=1}^{n} (N_i : M) \subseteq (P : M)$. Claim: $(N_j : M) \subseteq (P : M)$ for some $j$. Suppose that $(N_j : M) \nsubseteq (P : M)$ for all $j$. So there exists $s_j \in (N_j : M)$ such that $s_j \notin (P : M)$ for all $j$...(1). Then $s = s_1^n s_2 s_3 \ldots s_n \in (N_j : M)$ for all $j \Rightarrow s_1^n s_2 s_3 \ldots s_n = s \in \bigcap_{j=1}^{n} (N_j : M) \subseteq (P : M)$...(2). Since $P$ is prime ternary subsemimodule, by Theorem 2.3, $(P : M)$ is prime ideal of $R$. Hence by (2), $s_j \in (P : M)$ for some $j$, a contradiction to (1). Now $(N_j : M) \subseteq (P : M)$ for some $j$. Hence $(N_j : M)RM \subseteq (P : M)RM \Rightarrow N_j \subseteq P$, since by Lemma 2.6. Converse is trivial.

 

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