International Research Journal of Pure Algebra-5(1), 2015, 7-11
Available online through www.rjpa.info ISSN 2248–9037

A THEORETICAL INVARIANT OF TOPOLOGICAL SPACE GRAPH

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(Received On: 31-12-14; Revised & Accepted On: 24-01-15)

ABSTRACT

Topological graphs are play an important role in the networking especially in virtual networking in computer science as well as in biological sciences. In this paper, we define a graph "Given a topological space τ on a non -empty set X, a graph $G_X(\tau)$ whose vertex set τ arcs exist between two elements of τ if one is included in another" is named as " Topological Space Graph". Also studied the concepts spanning star, valency and obtained some results on these concepts.

Key words: *Graph, Topological graph, Topological space graph, valency.*

1. INTRODUCTION

Topological graphs are play an important role in the networking especially in virtual networking in computer science as well as in biological sciences. Based on the above fact, myself motivated to work on this way which shows a novel approach to define a topological space graph using elementary topology aspects. This work also shows the some results on valency of nodes of varies kind of this graphs.

Topological spaces: A topology T on a set X is a family of subsets of X with the following properties:
(1) Any union of elements of T is in T.
(2) Any finite intersection of elements of T is in T.
(3) and X are in T .

We say that (X, T) is a topological space, or abbreviated "X is a topological space"

2. Preliminaries:

2.1 Definition: A graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge two vertices (not necessarily distinct) called its end points.

2.2 Example (Königsberg bridge problem): The city of Königsberg (now Kaliningrad) used to have seven bridges across the river, linking the banks with two islands. The people living in Königsberg had a game where they would try to walk across each bridge once and only once. You can choose where to start.

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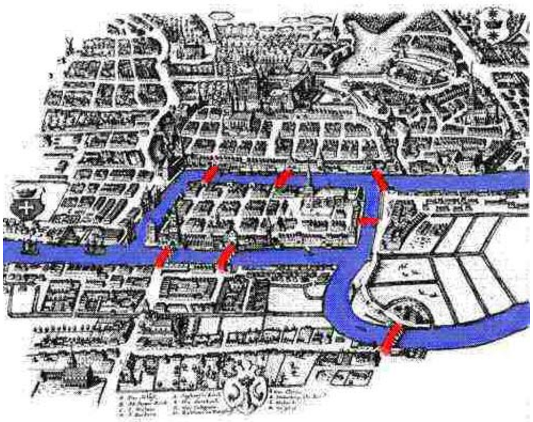
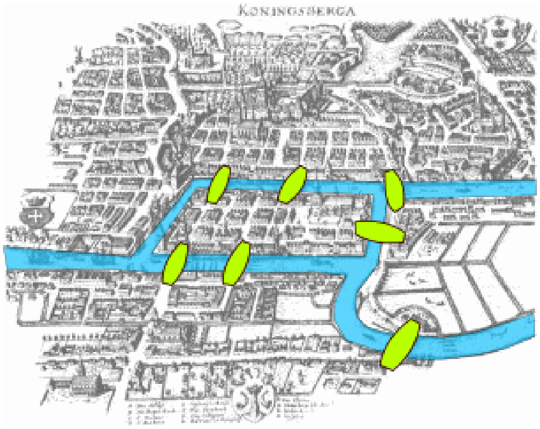
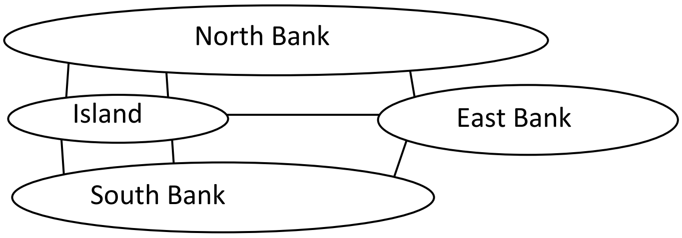


Solution: Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right[1].

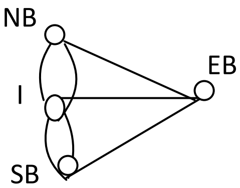


As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

2.3 Definitions:

(i) A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

(ii) A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle

(iii) A graph G is **connected** if each pair of vertices in G belongs to a path; otherwise, G is disconnected.

(iv) A graph is **finite** if its vertex set and edge set are finite.

We adopt the convention that every graph mentioned in this paper is finite, unless explicitly constructed otherwise.

2.4 Definition: A graph G is **bipartite** if V (G) is the union of two disjoint (possibly empty) independent sets called partite sets of G.

2.5 Notation: The (unlabeled) path and cycle with n vertices are denoted P_n and C_n respectively; an n- cycle is a with n vertices.

2.6 Definitions: (i) A **complete graph** is a simple graph whose vertices are pair wise adjacent; the (unlabeled) complete graph with n vertices is denoted K_n.
(ii) A **complete bipartite graph** or biclique is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes r and s, the (unlabeled) bi clique is denoted $K_{r, s}$.

2.7 Definition: A **subgraph** of a graph G is a graph H such that V(H) $\subseteq$ V(G) and E (H) $\subseteq$ E (G) and the assignment of endpoints to edges in H is the same as in G. We then write H $\subseteq$ G and say that "G contains H".

We refer the reader to [1, 2] for basic definitions and facts concerning topological spaces and graph theory. We write X for the non-empty set and, given a topological space T and two distinct elements $u, v \in T$ we denote $G_X(\tau)$ for the topological space graph on a topological space (X, T), and τ the vertex set of the topological space graph $G_X(\tau)$.

The following lemma shows some obvious observations whose verification (partly indicated in the lemma) is not present here which is collected from any elementary set theory books.

2.8. Lemma

(i) Let u, v be any two subsets of a universal set X. Then $u \cup v = u \;\; or\; v$
$\Leftrightarrow u \cap v = v\; or\; u.$
(ii) Let u, v be any two subsets of a universal set X. Then $u \subseteq v \;\Leftrightarrow u \cup v = v$
$\Leftrightarrow u \cap v = u.$
(iii) Let u be any subset of a universal set X. Then $\emptyset \subseteq u$ & $u \subseteq X.$

Proof this lemma is straight forward. So we left the proof for the readers.

3. The topological space graph $G_X(\tau)$

In this section studied about valency, Maximam and minimum valency of a graph, and we introduced the topological space graph $G_X(\tau)$, obtained some results on this concept.

We denote the graph with vertex set τ , edge set $E\left(G_X(\tau)\right)$ and τ the topological space on this set X .

We are also proved some results on maximum and minimum valency of a vertices of the graph.

3.1 Definitions:

(i) Let X be a non-empty set, τ be an arbitrary topology on X. A graph $G_X(\tau)$ whose vertex set is τ and two vertices u, v of $G_X(\tau)$ are adjacent if and only if $u \cup v = u \; or\, v$. This graph is called "**Topological space graph"** .It is denoted by $G_X(\tau)$.

(ii) Let X be the set of vertices and ***E*** the set of ***edges*** of G. For each x ∈ X, ***d(x)*** is the number of edges containing x. We call ***d(x)*** the degree of the vertex. Sometimes it is also known as **valency** of the vertex x.

The maximum degree of a vertex is also called a **maximum valency** of that vertex and minimum degree is called **minimum valency** of that vertex of a topological space graphs.

3.1. Result: $< u, v >$ is an edge in $G_X(\tau)$ if and only if $u \cap v = u \; or \; v$

Proof: Since from lemma 2.8, we have: $u \cup v = u \;\; or\; v \;\Leftrightarrow u \cap v = v\; or\; u$
$< u, v >$ is an edge in $G_X(\tau)$ $\Rightarrow u \cup v = u\, or\, v$

$\Rightarrow u \subseteq v \, or \, u \subseteq v$
$\Rightarrow u \cap v = u \, o \quad ur \cap v = u$
$\Rightarrow u \cap v = u \, or \; v$.

Conversely, let $u \cap v = u \; or \; v$
$\Rightarrow u \subseteq v \, or \, v \subseteq u$
$\Rightarrow u \cup v = u \, or \, v \cup u = v$
$\Rightarrow u \cup v = u \, or \, v$
$\Rightarrow < u, v >$ is an edge in $G_X(\tau)$.

3.2. Result: **T**he maximum valency of $G_X(\tau)$ is |τ| - 1.

Proof: since the vertex set T of the Graph $G_X(\tau)$ is a topology, so ϕ, X are two vertices of this graph $G_X(\tau)$
$\Rightarrow \phi \cup u = u$, for all $u \in \tau$ and $X \cup u = X$, for all $u \in \tau$
$\Rightarrow < \phi, u>$ and $<X, u>$, for all $u \in \tau$.

Then $G_X(\tau)$ = deg(ϕ) = deg(X) = |τ| - 1.

Note that vertices of $G_X(\tau)$ other than ϕ are may or may not satisfies the condition, $u \cup v = u \, or \, v$

So, $\deg(u) \leq$ |τ| - 1, for all $u \in \tau - \{\phi, X\}$.

Thus, The maximum valency of a vertices in $G_X(\tau)$ is |τ| - 1.
$\Rightarrow \phi \cup u = u, \forall u \in T \;\&\; X \cup u = X, \forall u \in T$

3.3. Result: The minimum valency of a topological space graph $G_X(\tau)$ is greater than or equal to two if the vertex set τ is a non-trivial topology on X.

Proof: Since T is non-trivial topology on X, order of τ is greater than or equal to 3 and it contains ∅ .

Any vertex u of $G_X(\tau)$ other than $X, \emptyset$, valency at least two, because $v \cup \phi = v$ and $v \cup X = X$ for every $v \in \tau$.

Thus valency of any vertex in $G_X(\tau)$ is greater than or equal to 2.

Hence the minimum valency of a topology graph $G_X(\tau)$ is greater than or equal to two if the vertex set T is a non-trivial topology on X.

3.4. Result: Any topological space graph $G_X(\tau)$ is non-empty

Proof: since vertex set in $G_X(\tau)$ is a topology τ
$\Rightarrow$ τ contain at least two elements say ϕ, X
$\Rightarrow < \phi, X >$ is an edge in $G_X(\tau)$ $(\because \phi \cup X = X)$
$\Rightarrow G_X(\tau)$ contains at least one edge
$\Rightarrow G_X(\tau)$ is non- empty.

3.5. Result: Any topological space graph $G_X(\tau)$ is simple

Proof: Let u, v are two vertices in $G_X(\tau)$.

According to set theory there exists two possibilities between u, v

Case-1: suppose that $u \cup v = u\, or\, v$. Since $u \cup v = u\, or\, v$ satisfies the commutative law,

So <u, v> = <v, u>

Thus $u \cup v = u\, or\, v \Rightarrow u, v$ have a unique edge in $G_X(\tau)$

Case-2: suppose that $u \cup v \neq u\, or\, v$.
Since $u \cup v \neq u\, or\, v \Rightarrow u, v$ have no edge in $G_X(\tau)$.
According to definition of $G_X(\tau)$, $< u, v >$ in $G_X(\tau)$ then $u \neq v$, so it have no loops .

3.6. Result: In any topological graph $G_X(\tau)$ where τ is a non-trivial topology, the distance between any two nodes is at most two.

Proof: Let u, v are two vertices in $G_X(\tau)$.
According to set theory there exists two possibilities between u, v ,

Case-1: suppose that $u \cup v = u\, or\, v$. So, $< u, v >$ is an edge in $G_X(\tau) \Rightarrow d(u, v) = 1$

Case-2: suppose $u \cup v \neq u\, or\, v$. So, $< u, v >$ is not an edge in $G_X(\tau)$ we now from lemma 2.8, u, v are adjacent to $\phi\ or\ X$ in $G_X(\tau) \Rightarrow d(u, v) = 2$. Thus, the distance between any two vertices in $G_X(\tau)$ is at most two.

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Source of Support: Nil, Conflict of interest: None Declared