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LINEAR PRESERVER OF IDEMPOTENTS OF TENSOR PRODUCTS ON MATRIX SPACE

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ABSTRACT

Let F be a field of characteristic not 2, M_n be the set consisting of all $n\times n$ matrices over F and P_n be the subset of M_n consisting of all idempotent matrices. We say that a linear map $\varphi: M_m\otimes M_n \to M_r$ preserves idempotents of tensor products of matrices, if for any $A\in M_m$ and $B\in M_n$, $\varphi(A\otimes B)$ is an idempotent whenever $A\otimes B$ is an idempotent. In this paper, we characterize linear maps $\phi: M_2\otimes M_2 \to M_r$, where ϕ preserves idempotents of tensor products of matrices.

***Key words**: Linear preserver, tensor product, idempotent.*

1. INTRODUCTION AND THE MAIN THEOREM

Let F be a field of characteristic not 2, M_n be the set consisting of all $n\times n$ matrices over F and P_n be the subset of M_n consisting of all idempotent matrices. For any $A=(a_{ij})\in M_m$ and $B\in M_n$, $A\otimes B$ denotes their tensor product, i.e., $A\otimes B=(a_{ij}B)$. I denotes the identity matrix and E_{ij} denote the matrix whose all entries are equal to zero except for the $(ij)-$th entry which is equal to 1 (Note: the dimension of I and E_{ij} can be known from the content). We say that a linear map $\varphi: M_m \to M_n$ preserves idempotents, if $\varphi(A)$ is an idempotent when A is an idempotent. We say that a linear map $\phi: M_{m_1}\otimes\cdots\otimes M_{m_l}\to M_n$ preserves idempotents of tensor products of matrices, if $\phi(A_1\otimes\cdots\otimes A_l)$ is an idempotent when $A_1\otimes\cdots\otimes A_l$ is an idempotent for any $A_1\otimes\cdots\otimes A_l\in M_{m_1}\otimes\cdots\otimes M_{m_l}$ .Especially, we call a linear map $\pi: M_{m_1\cdots m_l}\to M_{m_1\cdots m_l}$ canonical, if $\pi(A_1\otimes\cdots\otimes A_l)=\tau_1(A_1)\otimes\cdots\otimes\tau_l(A_l)$, where $\tau_k: M_{m_k}\to M_{m_k}$ is either the identity map $X\to X$ or the transposition map $X\to X^t, k=1,\cdots,l.$

This paper studied linear preserver problems on matrix spaces. In recent years, some preserver problems related tensor product were produced with some background in quantum information science. Some researchers have studied in many papers, see [1-3]. Especially, in connection to the linear preservers of idempotents, Zheng, Xu and Fošner [4] studied linear maps preserve idempotents of tensor products of matrices, they got the following theorem:

Theorem 1.1: Suppose l,n and $m_1,\cdots,m_l>2$ are positive integers with $n\le m_1\cdots m_l$. A linear map $\phi: M_{m_1}\otimes\cdots\otimes M_{m_l}\to M_n$ preserves idempotents of tensor products of matrices if and only if either $\phi=0$ or $n=m_1\cdots m_l$ and there is an invertible matrix $Q\in M_n$ and a canonical map π on $M_{m_1\cdots m_l}$ such that ϕ has the form

$$\phi(A_1\otimes\cdots\otimes A_l)=Q\pi(A_1\otimes\cdots\otimes A_l)Q^{-1} \quad (1)$$

for all $A_k\in M_{m_k}, k=1,\cdots,l.$

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Obviously, the case of $n > m_1 \cdots m_l$ is much more complex. So, we tried to study the problem in a comparatively simple case, i.e., in the case of $l = 2$ and $m_1 = m_2 = 2.$ we obtain that:

Theorem 1.2: Suppose n is a positive integer with $n > 4.$ A linear map $\phi: M_{m_2} \otimes M_{m_2} \to M_n$ preserves idempotents of tensor products of matrices if and only if there is an invertible matrix $Q \in M_n$ and natural numbers p_1, p_2, p_3, p_4 such that ϕ has the form

$$\phi(A \otimes B) = Q(A \otimes B \otimes I_{p_1} \oplus A \otimes B^t \otimes I_{p_2} \oplus A^t \otimes B \otimes I_{p_3} \oplus A^t \otimes B^t \otimes I_{p_4} \oplus 0)Q^{-1} \quad (2)$$

for all $A, B \in M_2$.

2. PRELIMINARY RESULTS

Lemma 2.1: Let m and n be positive integers. A linear map $\phi: M_m \to M_n$ preserves idempotents if and only if there exist an invertible $Q \in M_n$ and natural numbers p, q such that $\phi(A) = Q(A \otimes I_p \oplus A \otimes I_q)Q^{-1},\ A \in M_m.$

In a similar way as [4], we can get the following two lemmas.

Lemma 2.2: If $P \in P_2$ and $\phi(P \otimes I_2) = 0_S \oplus I_r \oplus 0_{n-r-s}$ for some nonnegative integers r and s, then there exists a linear map $\psi: M_2 \to M_r$ such that

$$\phi(P \otimes X) = 0_s \oplus \psi(X) \oplus 0_{n-r-s}$$

for all $X \in M_2$.Moreover, ψ preserves idempotents.

Lemma 2.3: Suppose that F_1 and F_2 are idempotents with $F_1 + F_2 \in P_2$.If rank $\phi(F_i \otimes I_2) = r$, $i = 1, 2,$ then there exist an invertible matrix $Q \in M_n$ and linear maps $g_i: M_2 \to M_r$ such that

$$\phi(F_i \otimes X) = Q(E_{ii} \otimes g_i(X) \oplus 0)Q^{-1},\ X \in M_2,\ i = 1,2.$$

Moreover, maps g_i preserve idempotents and $g_i(I) = I_r$.

Lemma 2.4: Suppose $p, q, p^{'}, q^{'}$ are natural integers, $U \in M_r$,

$$r = 2(p+q) = 2(p^{'} + q^{'}), \quad (3)$$

and $f(X) = X \otimes I_p \oplus X^t \otimes I_q, g(X) = X \otimes I_{p^{'}} \oplus X^t \otimes I_{p^{'}}, \forall X \in M_2$. If

$$f(X)U = Ug(X), \forall X \in M_2, \quad (4)$$

then $U = I_2 \otimes U_1 \oplus I_2 \otimes U_2$, where $U_1 \in M_{p \times p^{'}}, U_2 \in M_{q \times q^{'}}$.

Moreover, if U is invertible, then $f(X) = g(X)$ and U_1, U_2 are invertible.

Proof: Let $U = [U_{ij}]_{4\times 4}$ with $U_{kk} \in M_{p\times p^{'}}, k = 1,2, U_u \in M_{q\times q^{'}}, l = 3,4.$ Let $X = E_{ij}$, i, j =1,2, respectively, then it follows from (4) that $U = I_2 \otimes U_1 \oplus I_2 \otimes U_2$, $U_1 \in M_{p\times p^{'}}, U_2 \in M_{q\times q^{'}}$.

If $p \neq p^{'}$, without loss of generality, we may suppose that $p \prec p^{'}$, then it follows from (3) that $q \succ q^{'}$. Thus $r = rankU \leq 2(p + q^{'}) \prec 2(q + q^{'}) = r$. It is a contradiction. So, $p = p^{'}, q = q^{'}$. Thus $f(X) = g(X)$ and U_1, U_2 are invertible.

Proof of the main theorem: The sufficiency part of the Main Theorem is clear, we consider only the necessity part.

Let $f: M_2 \to M_n$; $A \mapsto \phi(A \otimes I)$, $\forall A \in M_2$.Then f is a linear map that maps idempotent matrices into idempotent matrices. By Lemma 2.1. we conclude that there exist an invertible $Q \in M_n$ and $p, q \in N$ such that

$$\phi(A \otimes I_2) = f(A) = Q(A \otimes I_p \oplus A^t \otimes I_q \oplus O)Q^{-1}, \quad A \in M_2. \tag{5}$$

Denote $r = p + q$. Then

$$rank\phi(E_{ij} \otimes I_2) = rank\phi((E_{ii} + \lambda E_{ij}) \otimes I_2) = r,\ i, j = 1,2,\ i \neq j,\ \forall \lambda \in F. \tag{6}$$

Composing ϕ by the appropriate similarity transformation, without loss of generality, we may assume that

$$\phi(E_{ii} \otimes I_2) = E_{ii} \otimes I_r \oplus O,\ i = 1, 2. \tag{7}$$

Thus, by Lemma 2.3 there exist linear maps $f_i : M_2 \to M_r$ such that

$$\phi(E_{ii} \otimes X) = E_{ii} \otimes f_i(X) \oplus 0,\ i = 1, 2, \tag{8}$$

and f_i preserve idempotents, i =1,2. In view of (7), using Lemma 2.1, without loss of generality, we can assume

$$f_i(X) = X \otimes I_{p_i} \oplus X^t \otimes I_{q_i},\ i = 1, 2. \tag{9}$$

Let $F_1 := E_{11} + E_{12}, F_2 := E_{22} - E_{12}$.Then, by (6) and Lemma 2.3, there exists an invertible $Q \in M_n$ and linear maps $g_i : M_2 \to M_r$ such that

$$\phi(F_i \otimes X) = Q(E_{ii} \otimes g_i(X) \oplus 0)Q^{-1}, \tag{10}$$

where g_i preserve idempotents and $g_i(I) = I_r$, i =1,2. Using Lemma 2.1, without loss of generality, we can assume

$$g_i(X) = X \otimes I_{p_i^{'}} \oplus X^t \otimes I_{q_i^{'}},\ i = 1, 2. \tag{11}$$

Let $Q = [Q_{ij}]_{3\times 3}$ with $Q_{ij} \in M_r, i, j$ =1,2 and $Q^{-1} = [T_{ij}]_{3\times 3}$ with $T_{ij} \in M_r, i, j$ =1,2.

Since $E_{12} \otimes X = F_1 \otimes X - E_{11} \otimes X$, by (8) and (10) we have

$$\phi(E_{12} \otimes X) = \begin{bmatrix} Q_{11}g_1(X)T_{11} - f_1(X) & Q_{11}g_1(X)T_{12} & Q_{11}g_1(X)T_{13} \\ Q_{21}g_1(X)T_{11} & Q_{21}g_1(X)T_{12} & Q_{21}g_1(X)T_{13} \\ Q_{31}g_1(X)T_{11} & Q_{31}g_1(X)T_{12} & Q_{31}g_1(X)T_{13} \end{bmatrix} \tag{12}$$

Similarly, since $E_{12} \otimes X = -(F_2 \otimes X - E_{22} \otimes X)$, by (8) and (10) we have

$$\phi(E_{12} \otimes X) = \begin{bmatrix} -Q_{12}g_2(X)T_{21} & -Q_{12}g_2(X)T_{22} & -Q_{12}g_2(X)T_{23} \\ -Q_{22}g_2(X)T_{21} & f_2(X) - Q_{22}g_2(X)T_{22} & -Q_{22}g_2(X)T_{23} \\ -Q_{32}g_2(X)T_{21} & -Q_{32}g_2(X)T_{22} & -Q_{32}g_2(X)T_{23} \end{bmatrix} \tag{13}$$

For any idempotent $P \in P_2$ and any scalar $\lambda \in F$, $(E_{ii} + \lambda E_{12}) \otimes P \in M_{2\times 2}$ is an idempotent, i =1,2. Then,

$$\phi(E_{ii} \otimes P) + \lambda \phi(E_{12} \otimes P) \in M_n,\ i = 1, 2.$$

Thus

$$\phi(E_{ii} \otimes P)\phi(E_{12} \otimes P) + \phi(E_{12} \otimes P)\phi(E_{ii} \otimes P) = \phi(E_{12} \otimes P),\ i = 1, 2 \tag{14}$$

and

$$[\phi(E_{12} \otimes P)]^2 = 0. \tag{15}$$

Using (8), (12)-(15), we obtain

$$\begin{cases} Q_{11}g_1(P)T_{11} = f_1(P),\ Q_{22}g_2(P)T_{22} = f_2(P) \\ Q_{21}g_1(P)T_{12} = 0,\ Q_{21}g_1(P)T_{13} = 0 \\ Q_{31}g_1(P)T_{12} = 0,\ Q_{31}g_1(P)T_{23} = 0 \\ -Q_{12}g_2(P)T_{21} = 0, -Q_{12}g_2(P)T_{23} = 0 \\ -Q_{32}g_2(P)T_{21} = 0, -Q_{32}g_2(P)T_{23} = 0 \end{cases}$$

Since any matrix of M_2 can be written as a linear combination of idempotents, we get

$$\begin{cases} Q_{11}g_1(X)T_{11} = f_1(X),\ Q_{22}g_2(X)T_{22} = f_2(X) \\ Q_{21}g_1(X)T_{12} = 0,\ Q_{21}g_1(X)T_{13} = 0 \\ Q_{31}g_1(X)T_{12} = 0,\ -Q_{31}g_1(X)T_{13} = 0 \\ -Q_{12}g_2(X)T_{21} = 0,\ -Q_{12}g_2(X)T_{23} = 0 \\ -Q_{32}g_2(X)T_{21} = 0,\ -Q_{32}g_2(X)T_{23} = 0 \end{cases} \tag{16}$$

Using (9), (11)-(16) and Lemma 2.4 we can conclude that $f_i(X) = g_i(X)$, i =1, 2 and

$$\phi(E_{12} \otimes X) = \begin{bmatrix} 0 & g_1(X)U & 0 \\ Vg_1(X) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Ug_2(X) & 0 \\ g_2(X)V & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

where

$U = Q_{11}T_{12} = -Q_{12}T_{22}$, $V = Q_{21}T_{11} = -Q_{22}T_{21}$.So, by Lemma 2.4 we get

$$U = I_2 \otimes U_1 \oplus I_2 \otimes U_2, V = I_2 \otimes V_1 \oplus I_2 \otimes V_2, \tag{18}$$

where $U_1 \in M_{p_1 \times p_2}$, $U_2 \in M_{q_1 \times q_2}$, $V_1 \in M_{p_2 \times p_1}$, $V_2 \in M_{q_2 \times q_1}$. Moreover, by (15) we can get $U_iV_i = 0,\ V_iU_i = 0,$ i =1, 2. This, together with (6) we can conclude that there exist invertible matrices P_1 and P_2 such that

$$U_i = P_1 \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix} P_2,\ V_i = P_2^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{s_i} \end{bmatrix} P_1^{-1},\ r_i + s_i \le \min\{p_i, q_i\},\ i=1,2. \tag{19}$$

In a similar way as above we can get

$$\phi(E_{21} \otimes X) = \begin{bmatrix} 0 & g_1(X)U' & 0 \\ V'g_1(X) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & U'g_2(X) & 0 \\ g_2(X)V' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

where

$$U' = I_2 \otimes U_1' \oplus I_2 \otimes U_2',\ V' = I_2 \otimes V_1' \oplus I_2 \otimes V_2'. \tag{21}$$

$$U_i'V_i' = 0,\ V_i'U_i' = 0,\ i=1,2. \tag{22}$$

and $U_1' \in M_{p_1 \times p_2}$, $U_2' \in M_{q_1 \times q_2}$, $V_1' \in M_{p_2 \times p_1}$, $V_2' \in M_{q_2 \times q_1}$.

For any idempotent $P \in P_2$, $\frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22}) \otimes P$ is idempotent, then $\frac{1}{2}\phi((E_{11} + E_{12} + E_{21} + E_{22}) \otimes P)$ is idempotent. Using this fact and (8), (9), (17)-(22) we conclude that $r_1 + s_1 = p_1 = p_2$, $r_2 + s_2 = q_1 = q_2$ and there exist invertible matrices Q_1 and Q_2 such that

$$U_i = Q_1\begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}Q_2,\ V_i = Q_2^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{s_i} \end{bmatrix}Q_1^{-1},\ i=1, 2.$$

$$U_i^{'} = Q_1\begin{bmatrix} 0 & 0 \\ 0 & I_{s_i} \end{bmatrix}Q_2,\ V_i^{'} = Q_2^{-1}\begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}Q_1^{-1},\ i=1, 2.$$

Thus $g_1(X) = g_2(X)$.

At last, composing ϕ by the appropriate similarity transformation again, we obtain

$$\phi(E_{ij} \otimes X) = E_{ij} \otimes X \otimes I_{r_1} \oplus E_{ij} \otimes X^t \otimes I_{r_2} \oplus E_{ji} \otimes X \otimes I_{s_1} \oplus E_{ji} \otimes X^t \otimes I_{s_2} \oplus 0,\ i=1, 2.$$

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