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COMMON FIXED POINT THEOREMS FOR EXPANDING MAPPINGS, WITH EXPANSION FACTOR CONTROLLED BY A NON DECREASING FUNCTION

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ABSTRACT

In this paper, we prove expansion mapping theorems in metric spaces the expansion factor being controlled by a non-decreasing function using the concept of compatible maps, weakly reciprocal continuity, R- weakly commuting of type (A f) and (P).

Keywords: *Expansion mapping, φ- weakly expansive mapping, compatible mapping, R- weakly commuting mapping, expanding maps, expansion factor.*

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1. INTRODUCTION AND PRELIMINARIES

In 1922, Banach proved the Banach contraction principle. Many authors have extended, generalized the Banach contraction Principle in different ways. In 1992, Daffer and Kaneko [2] proved a fixed point theorem for expansive mappings.

Definition 1.1: Let f *be* a self -mapping of a metric space (X, d). Then f *is* said to be expansive if there exists a real number $h > 1$ such that

$$d(f x,\ f y) \geq h\, d(x, y) \text{ for all } x, y \in X. \tag{1.1.1}$$

In 1997, Alber and Gurre - Delabrire [1] introduced the notion of φ- weak contraction as follows.

Definition 1.2: [1] Let f be a self - mapping of a metric space (X, d). Then f is said to be φ - weak contraction if there exists a continuous mapping $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$ such that

$$d (f (x),\ f (y)) \leq d(x, y) - \varphi(d(x, y)) \text{ for all } x, y \in X. \tag{1.2.1}$$

Recently S. M. Kang, M. Kumar, P.Kumar and S. Kumar [5] introduced φ-weakly expansive mappings in metric spaces as follows.

Definition 1.3: [5] Let f be a self -mapping of a metric space (X, d). Then f is said to be φ - weakly expansive if there exists a continuous mapping $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi (0) = 0$ and $\varphi (t) > t$ for all $t > 0$ such that

$$d (f x,\ f y) \geq d(x, y) + \varphi(d(x, y)) \text{ for all } x, y \in X. \tag{1.3.1}$$

Definition 1.4: [5] Let f and g be two self- mappings of a metric space (X, d). Then f is said to be φ - weakly expansive with respect to $g\colon X \to X$ *if* there exists a continuous mapping φ: $[0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > t$ for all $t > 0$ such that

$$d(f x,\ f y) \geq d(gx, gy) + \varphi(d(gx, gy)) \text{ for all } x, y \in X. \tag{1.4.1}$$

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In 1986, Jungck [4] defined the concept of compatible mappings.

Definition 1.5: [4] A pair (f, g) of self-mappings of a metric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, g fx_n) = 0$, whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some $z \in X$.

In 1994, Pant [8] introduced the notion of point wise R- weak commutativity in metric spaces.

Definition 1.6: [8] Let f and g be two self -mappings of a metric space (X, d). Then f and g are called R-weakly commuting on X if there exists $R > 0$ such that

$$d(fgx, gfx) \le Rd(fx, gx) \quad \text{for all } x \in X. \tag{1.6.1}$$

It is obvious that R-weakly commuting mappings commute at their coincidence points and hence R-weak commutativity is equivalent to commutativity at coincidence points.

In 1997, Pathak *et al.* [12] generalized the notion of R-weakly commuting mappings to R-weakly commuting mappings of type (Ag) and of type (Af).

Definition 1.7: [12] Let f and g be two self -mappings of a metric space (X, d). Then f and g are called R-weakly commuting of type (Ag) if there exists $R > 0$ such that

$$d(ffx, gfx) \le Rd(fx, gx) \text{ for all } x \in X.$$

Similarly, the two self -mappings f and g are called R-weakly commuting of type (Af) if there exists $R > 0$ such that $d(fgx, ggx) \le Rd(fx, gx)$ for all $x \in X$.

Definition 1.8: [12] Let f and g be two self -mappings of a metric space (X, d). Then f and g are called R -weakly commuting of type (P) if there exists $R > 0$ such that

$$d(ffx, ggx) \le Rd(fx, gx) \text{ for all } x \in X.$$

In 1998, [10] introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in a metric space. The notion of reciprocal continuity is weaker than the continuity of one of the mappings.

Definition 1.9: [10] Two self mappings f and g of a metric space (X, d) are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

In 2011, Pant *et al.* [11] generalized the notion of reciprocal continuity to weak reciprocal continuity as follows.

Definition 1.10: [11] Two self mappings f and g of a metric space (X, d) are called weakly reciprocally continuous if $fgx_n = fz$ or $\lim_{n\to\infty} gf\ x_n = gz$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for some z in } X.$$

In 1992, Daffer and Kaneko [2] proved the following fixed point theorem.

Theorem 1.11: [2] Let (X, d) be a complete metric space. Let f be a surjective self map and g be an injective self map of X which satisfy the following conditions: There exists a number $q > 1$ such that

$$d(fx, fy) \ge q\, d(gx, gy) \text{ for each } x, y \text{ in } X,$$

then f and g have a unique common fixed point.

In 1993, B. E. Rhoades extended Theorem 1.11 to compatible mappings as follows.

Theorem 1.12: [13] Let (X, d) be a complete metric space. Let f and g be compatible self maps of X satisfying

(i) $gX \subseteq f\ X$;
(ii) there exists $q > 1$ such that $d(fx, fy) \ge q\, d(gx, gy)$ for each x, y in X, and
(iii) f is continuous.

Then f and g have a unique common fixed point.

In 2008, Kumar [6] generalized Theorem 1.12 to weakly compatible maps as follows.

Theorem 1.13: [6] Let (X, d) be a complete metric space. Let f and g be weakly compatible self maps of X satisfying

(i) $gX \subseteq fX$;
(ii) there exists $q > 1$ such that $d(fx, fy) \geq q\, d(gx, gy)$ for all $x, y \in X$.

If one of the subspaces gX or fX is complete, then f and g have a unique common fixed point.

In 2012, S. Manro and P. Kumar [7] proved the following theorem, using the concept of compatibility and weak reciprocal continuity in complete metric spaces.

Theorem 1.14: [7] Let f and g be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying

(i) $gX \subseteq fX$;
(ii) there exists $q > 1$ such that $d(fx, fy) \geq q\, d(gx, gy)$ for all $x, y \in X$,

If f and g are either compatible or R- weakly commuting of type (Ag) or, R- weakly commuting of type (Af) or R-weakly commuting of type (P), then f and g have a unique common fixed point.

Recently S. M. Kang *et al.* [5] generalized and extended Theorem 1.4 for φ- weakly expansive mappings as follows.

Theorem 1.15: [5] Let f and g be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying

(i) $gX \subseteq fX$;
(ii) there exists a continuous mapping $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > t$ for all $t > 0$ such that $d(fx, fy) \geq d(gx, gy) + \varphi(d(gx, gy))$ for all $x, y \in X$, and if f and g are compatible, then f and g have a unique common fixed point.

Theorem 1.16: [5] Let f and g be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying

(i) $gX \subseteq fX$;
(ii) there exists a continuous mapping $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > t$ f or all $t > 0$ such that $d(fx, fy) \geq d(gx, gy) + \varphi(d(gx, gy))$ for all $x, y \in X$, and if f and g are R - weakly commuting of type (Ag) or, R- weakly commuting of type (Af) or R - weakly commuting of type (P), then f and g have a unique common fixed point.

We observe that in Theorem 1.15 and Theorem 1.16 the condition '$\varphi(t) > 0$ for all $t > 0$' is unnecessary. Further, we obtain common fixed point theorems when φ is non-decreasing (but not necessarily continuous).

An example also is provided in support of our result.

2. MAIN RESULTS

We begin with some definitions.

Definition 2.1: A function $\varphi: [0, \infty) \to [0, \infty)$ is called a control function if

(i) φ is non-decreasing and
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.2: Suppose (X, d) is a metric space and f, g are two self maps on X. Suppose φ is a control function such that

$$d(fx, fy) \geq d(gx, gy) + \varphi(d(gx, gy)) \text{ for all } x, y \in X.$$

Then f *is* said to be expanding with respect to g with expansion factor $\varphi(d(gx, gy))$ for all $x, y \in X$.

Now we obtain conditions for the existence of a common fixed point for two self maps f and g on a complete metric space, when f is expanding with respect to g, the control function being φ.

The exact statement of the result is as follows.

Theorem 2.3: Let f and g be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying

(i) $g(X) \subseteq f(X)$;

(ii) there exists a mapping $\varphi : [0,\infty) \to [0,\infty)$ such that φ is non-decreasing and $\varphi(t) = 0$ if and only if $t = 0$ and $d(fx, fy) \geq d(gx, gy) + \varphi(d(gx, gy))$ for all $x, y \in X$. (2.3.1)

If f and g are compatible,

Then f and g have a unique common fixed point.

Proof: Let $x_0 \in X$.

Since $g(X) \subseteq f(X)$, we can choose $x_1 \in X$ such that $gx_0 = fx_1$.

In general we can choose $\{x_n\}$ in X such that $gx_n = fx_{n+1}$ for $n = 0, 1, 2, \ldots$.

Write $y_n = gx_n = f\ x_{n+1}$ (2.3.2)

If $y_n = y_{n+1}$ for some $n \in N$, then we have $gx_n = gx_{n+1}$ so that $gx_n = fx_{n+1} = gx_{n+1}$.

This implies that x_{n+1} is a coincidence point of f *and* g.

Since f and g are compatible, we have $fgx_{n+1} = gfx_{n+1}$ so that $fgx_n = ggx_n$ (2.3.3)

and hence gx_n is a coincidence point of f and g.

Now, from (2.3.1), we have

$$d(fx_{n+1}, fgx_n) \geq d(gx_{n+1}, ggx_n) + \varphi(d(gx_{n+1}, ggx_n)) \quad (2.3.4)$$
$$\geq d(fx_{n+1}, fgx_n) + \varphi(d(fx_{n+1}, ggx_n)) \quad (2.3.5)$$
$$0 \geq \varphi(d(fx_{n+1}, ggx_n))$$
$$0 \geq \varphi(d(fx_{n+1}, fgx_n))$$
$$0 = d(fx_{n+1}, fgx_n)$$
$$0 = d(gx_n, fgx_n)$$

That implies $gx_n = fgx_n$.

Therefore gx_n is a fixed point of f.

From (2.3.5) and (2.3.3), we have

$$0 \geq \varphi(d(gx_{n+1}, ggx_n))$$

therefore $0 \geq \varphi(d(fx_{n+1}, fgx_n))$

therefore $0 = d(fx_{n+1}, fgx_n))$

This implies $f\,x_{n+1} = fgx_n$

$fx_{n+1} = ggx_n$

$gx_n = ggx_n$ and hence gx_n is a fixed point of g.

Therefore gx_n is a common fixed point of f and g.

Hence we may assume that without loss of generality that $y_n \neq y_{n+1}$ for all $n \in N$

so that $d(y_n, y_{n+1}) > 0$ for all $n \in N$.

From (2.3.1), we have

$$d(y_n, y_{n-1}) = d(fx_{n+1}, fx_n)$$
$$\geq d(gx_{n+1}, gx_n) + \varphi(d(gx_{n+1}, gx_n))$$
$$= d(y_{n+1}, y_n) + \varphi(d(y_{n+1}, y_n)) \quad (2.3.6)$$
$$> d(y_{n+1}, y_n)$$

Therefore $d(y_{n+1}, y_n) < d(y_n, y_{n-1})$.

Thus the sequence $\{d(y_{n+1}, y_n)\}$ is a strictly decreasing sequence of positive real numbers and so $\lim\limits_{n\to\infty} d(y_{n+1}, y_n)$ exists and it is r (say). *i.e.*, $\lim\limits_{n\to\infty} d(y_{n+1}, y_n) = r \geq 0$. (2.3.7)

Now $d(y_{n+1}, y_n) < d(y_n, y_{n-1})$.

Since φ is non-decreasing we have $\varphi(d(y_{n+1}, y_n)) \leq \varphi(d(y_n, y_{n-1}))$.

Therefore the sequence $\varphi(d(yn_{+1}, y_n))$ is a decreasing sequence of nonnegative real's and so $\lim\limits_{n\to\infty} \varphi(d(y_{n+1}, y_n))$ exists and it is s (say).

i.e., $\lim\limits_{n\to\infty} \varphi(d(y_{n+1}, y_n)) = s \geq 0$. (2.3.8)

We now show that $r = 0$.

From (2.3.6), we have $d(y_n, y_{n-1}) \geq d(y_{n+1}, y_n) + \varphi(d(y_{n+1}, y_n))$.

On letting $n \to \infty$, from (2.3.7) and (2.3.8) we get $r \geq r + s$, so that $s = 0$.

Now $r \leq d(y_{n+1}, y_n)$.

Since φ is non- decreasing we have $\varphi(r) \leq \varphi(d(y_{n+1}, y_n))$ so that $\varphi(r) \leq l\lim\limits_{n\to\infty} \varphi(d(y_{n+1}, y_n)) = s = 0$.

That implies $\varphi(r) = 0$ so that $r = 0$. *i.e.*, $\lim\limits_{n\to\infty} d(y_{n+1}, y_n) = 0$

Now, we show that $\{y_n\}$ is Cauchy.

Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ and $d(ym(k), yn(k)) > \varepsilon$ and $d(y_{m(k)}, y_{n(k)-1}) \leq \varepsilon$.

The following identities can be established.

(*i*) $\lim\limits_{k\to\infty} d(y_m(k), y_n(k)) = \varepsilon$, (*ii*) $\lim\limits_{k\to\infty} d(y_m(k)\text{-}1, y_n(k)\text{-}1) = \varepsilon$,

Hence $d(y_{m(k),} y_{n(k)}) > \dfrac{\varepsilon}{2}$ for large k (2.3.9)

$d(y_{m(k)-1}, y_{n(k)-1}) = d(fx_{m(k)}, fx_{n(k)})$

$$\begin{aligned} d(y_{m(k)-1}, y_{n(k)-1}) &= d(f(x_{m(k)}, f(x_{n(k)}) \\ &\geq d(g_{xm(k)}, gx_{n(k)}) + \varphi(d(gx_{m(k)}, gx_{n(k)})) \\ &= d(y_{m(k)}, y_{n(k)}) + \varphi(d(y_{m(k)}, y_{n(k)})) \\ &\geq d(y_{m(k)}, y_{n(k)}) + \varphi\left(\frac{\varepsilon}{2}\right) \quad \text{(by (2.3.9))} \end{aligned}$$

On letting $k \to \infty$, we get

$\varepsilon \geq \varepsilon + \varphi\left(\dfrac{\varepsilon}{2}\right)$ that implies $\varphi\left(\dfrac{\varepsilon}{2}\right) = 0$ so that $\varepsilon = 0$, a contradiction.

Hence $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists a point $z \in X$ such that, $\lim\limits_{n\to\infty} y_n = z$.

Then by (2.3.2), we have $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} fx_{n+1} = z$.

Since f and g are compatible mappings, we have, $\lim\limits_{n\to\infty} \mathrm{d}(\mathrm{fgx_n}, \mathrm{gfx_n}) = 0$ (2.3.10)

Also, by the weak reciprocal continuity of f and g.

We have $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$.

Let $\lim_{n\to\infty} fgx_n = f z$.

From (2.3.10) $\lim_{n\to\infty} d(f z, g f x_n) = 0$, so that $\lim_{n\to\infty} gfx_n = fz$.

Now, we claim that $f z = gz$.

Let $fz \neq gz$.

From (2.3.2), $\lim_{n\to\infty} gf x_{n+1} = \lim_{n\to\infty} ggx_n = f z$.

By (2.3.1)
$d(fz, fgxn) \geq d(gz, ggx_n) + \varphi(d(gz, ggx_n))$
$\geq d(gz, ggxn)$.

On letting $n \to \infty$, we get
$d(fz, f z) \geq d(gz, f z)$

that implies $0 \geq d(gz, f z)$.

Hence $f z = gz$.

Therefore z is a coincidence point of f and g.

Since $fz = gz$, by the compatibility of f and g we have
$fgz = gfz = ggz$.

Consider
$d(gz, ggz) = d(fz, fgz)$
$\geq d(gz, ggz) + \varphi(d(gz, ggz))$
$0 \geq \varphi(d(gz, ggz))$

Therefore $0 = d(gz, ggz)$.

Therefore $gz = ggz$ and hence gz is a fixed point o f g.

Also we have $gz = ggz = fgz$ so that $gz = fgz$

and hence gz is a fixed point of f .

Therefore gz is a common fixed point of f and g.

When $\lim_{n\to\infty} gfx_n = gz$, we can prove the result in a similar way.

Uniqueness

Let u and v be two common fixed points of f and g.

From (2.3.1), we have
$d(u,v) = d(fu, fv)$
$\geq d(gu, gv) + \varphi(d(gu, gv))$
$= d(u, v) + \varphi(d(u, v))$
$0 = \varphi(d(u, v))$

so that $d(u, v) = 0$ and hence $u = v$.

Therefore f and g have a unique common fixed point.

Now, we prove a common fixed point theorem for a R- weakly Commuting of type (Af) or of type P.

Theorem 2.4: Let f and g be two weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying

(i) $gX \subseteq fX$

(ii) there exists a mapping $\varphi : [0,\infty) \to [0,\infty)$ such that φ is non-decreasing and $\varphi(t) = 0$ if and only if $t = 0$ and

(iii) $d(fx, fy) \geq d(gx, gy)+\varphi(d(gx, gy))$ for all $x, y \in X$. (2.4.1)

If f and g are R- weakly commuting of type (Af) or R- weakly commuting of type (P), then f and g have a unique common fixed point..

Proof: Let $\{x_n\}$ and $\{y_n\}$ be as in Theorem 2.3. Again from the proof of Theorem 2.3 it follows that, $\{yn\}$ is a Cauchy sequence in X.

Since X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Then by (2.3.2), we have

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n+1} = z$.

Now, suppose that f and g are R- weakly commuting of type (Af).

Then we have $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$ for all $x_n \in X$. (2.4.2)

Now, from the weak reciprocal continuity of f and g, we get that $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$.

Let $\lim_{n\to\infty} fgx_n = fz$.

From (2.4.2), we have $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$.

On letting $n \to \infty$, we get

$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq R \lim_{n\to\infty} d(fx_n, gx_n) = 0$

Therefore $\lim_{n\to\infty} ggx_n = fz$.

Now, we claim that $fz = gz$.

Let $fz \neq gz$. By (2.4.1)

$d(fz, fgx_n) \geq d(gz, ggx_n) + \varphi(d(gz, ggx_n))$

$\geq d(gz, ggx_n)$

On letting $n \to \infty$, we get

$d(fz, fz) \geq d(gz, fz)$

$0 \geq d(gz, fz)$.

Hence $gz = fz$.

Therefore z is a coincidence point of f and g.

Again by R- weak commutativity of type (Af), we have

$d(fgz, ggz) \leq Rd(gz, fz) = 0$.

Therefore $fgz = ggz$.

Now consider

$d(gz, ggz) = d(fz, fgz)$

$\geq d(gz, ggz) + \varphi(d(gz, ggz))$

$0 \geq \varphi(d(gz, ggz))$

$0 = d(gz, ggz)$

Therefore $gz = ggz$ and hence gz is a fixed point of g.

Also we have $gz = ggz = fgz$ which implies that $gz = fgz$ and hence gz is a fixed point of f.

Therefore gz is a common fixed point of f and g.

Similarly, if $\lim_{n\to\infty} gf\, x_n = gz$, we get that f and g have common fixed point.

Now, suppose that f and g are R- weakly commuting of type (P).

Then we have $d\,(ffx_n, ggx_n) \leq Rd(fx_n, gx_n)$ for all $x_n \in X$. (2.4.3)

Again, by the weak reciprocal continuity of f and g,

we have $\lim_{n\to\infty} fgx_n = f\,z$ or $\lim_{n\to\infty} gfx_n = gz$.

Let $\lim_{n\to\infty} fgx_n = f\,z$.

$\lim_{n\to\infty} (ffxn, ggx_n) \leq \lim_{n\to\infty} Rd\,(fxn, gx_n) = Rd(z, z) = 0$.

Therefore $\lim_{n\to\infty} d(ffx_n, ggx_n) = 0$.

Using (2.3.2), we have $f\,gx_{n-1} = f\,f\,x_n \to f\,z$ and $\lim_{n\to\infty} d\,(f\,z, ggx_n) = 0$ that implies $\lim_{n\to\infty} ggx_n = f\,z$.

Now, we claim that $f\,z = gz$.

Let $f\,z \neq gz$.

By (2.4.1), we have

$$d\,(fz, fgx_n) \geq d(gz, ggx_n) + \varphi(d(gz, ggx_n))$$
$$\geq d(gz, ggx_n)$$

On letting $n \to \infty$, we get

$$d(fz, fz) \geq d(gz, fz)$$
$$0 \geq d(gz, fz).$$

Hence $gz = fz$.

Therefore z is a coincidence point of f and g.

Again by R- weak commutativity of type (P), we have
$d(fgz, ggz) \leq Rd(gz, fz) = 0$.

Therefore $fgz = ggz$.

Therefore $f\,f\,z = fgz = ggz$.

Now consider

$$d(gz, ggz) = d(fz, fgz)$$
$$\geq d(gz, ggz) + \varphi(d(gz, ggz))$$
$$0 \geq \varphi(d(gz, ggz))$$
$$0 = d(gz, ggz)$$

Therefore $gz = ggz$ and hence gz is a fixed point of g.

Also we have $gz = ggz = fgz$. Thus $gz = fgz$ and hence gz is a fixed point of f.

Therefore gz is a common fixed point of f and g.

Similarly, if $\lim_{n\to\infty} g\,fx = gz$, we can easily prove that f and g have common fixed point.

Uniqueness follows as in Theorem 2.3.

In Theorem 2.3, if g is the identity mapping, then we obtain the following.

Theorem 2.5: Let f be a surjective self mapping of a complete metric space (X, d) satisfying

(i) there exists a mapping $\varphi : [0,\infty) \to [0,\infty)$ such that φ is non-decreasing and $\varphi(t) = 0$ if and only if $t = 0$ and

(ii) $d(fx, fy) \geq d(x, y) + \varphi(d(x, y))$ for all $x, y \in X$ (2.5.1)

.

Then f has a unique fixed point.

The following is a supporting example of Theorem 2.3 and Theorem 2.4.

Here φ non-decreasing but is neither continuous nor satisfies the Condition: $\varphi(t) > t$ for all $t > 0$.

Example 2.6: Let $X = [0, 1]$ be endowed with the usual metric.

We define $f, g: X \to X$ by $fx = \dfrac{x}{2}$ and $gx = \dfrac{x}{4}$ and define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \begin{cases} t & if\ 0 \leq t \leq \frac{1}{4} \\ 2t & if\ \ t > \frac{1}{4} \end{cases}$

Then $g(X) = [0, \frac{1}{4}] \subseteq [0, \frac{1}{2}] = f(X)$.

$d(fx, fy) = \left|\dfrac{x-y}{2}\right|$

$d(gx, gy) = \left|\dfrac{x-y}{4}\right|$

$\varphi(d(gx, gy)) = \left|\dfrac{x-y}{4}\right|$

$\left|\dfrac{x-y}{2}\right| = d(fx, fy) \geq d(gx, gy) + \varphi(d(gx, gy)) = \left|\dfrac{x-y}{2}\right|$ holds for all $x, y \in [0, 1]$, f and g satisfy all the conditions of Theorem 2.3 and Theorem 2.4 and 0 is the unique fixed point.

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