International Research Journal of Pure Algebra-5(3), 2015, 26-30
Available online through www.rjpa.info ISSN 2248–9037

THE NEIGHBORHOOD TOTAL EDGE DOMINATION NUMBER OF A GRAPH

V. R. Kulli*

Department of Mathematics, Gulbarga University, Gulbarga 585 106, India.

(Received On: 20-02-15; Revised & Accepted On: 19-03-15)

ABSTRACT

Let $G = (V, E)$ be a graph without isolated vertices and isolated edges. An edge dominating set F of G is called a neighborhood total edge dominating set if the edge induced subgraph $\langle N(F)\rangle$ has no isolated edges. The neighborhood total edge domination number $\gamma'_{nt}(G)$ of G is the minimum cardinality of neighborhood total edge dominating set of G. In this paper, we initiate a study of this new parameter.

Keywords: *edge domination, connected edge domination, total edge domination, neighborhood total edge domination.*

Mathematics subject classification: *05C69.*

1. INTRODUCTION

All graphs considered here are finite, undirected without loops and multiple edges. Unless and otherwise stated, the graph $G = (V, E)$ considered here have $p = |V|$ vertices and $q = |E|$ edges. Any undefined term in this paper may be found in Kulli [2].

A set D of vertices in a graph G is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently several domination parameters are given in the books by Kulli [3,4, 5].

A set E of edges in a graph G is called an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set of G. The concept of edge domination was introduced by Mitchell and Hedetniemi in [21] and was studied by several authors for example [1, 6, 9, 10, 11, 12, 20].

An edge dominating set F of a graph G is a connected edge dominating set if the edge induced subgraph $\langle F\rangle$ is connected. The connected edge domination number $\gamma'_c(G)$ of G is the minimum cardinality of a connected edge dominating set of G. The concept of connected edge domination was introduced by Kulli and Sigarkanti in [16] and was studied in [17]. A set F of edges in a graph $G = (V, E)$ is called a total edge dominating set of G if every edge in E is adjacent to at least one edge in F. The total edge domination number $\gamma'_t(G)$ of G is the minimum cardinality of a total edge dominating set of G. This concept was introduced by Kulli and Patwari in [15] and was studied for example [7, 8].

The vertices and edges of a graph G are called the elements of G. A set X of elements of G is an entire dominating set if every element not in X is either adjacent or incident to at least one element in X. The entire domination number $\varepsilon(G)$ of G is the minimum cardinality of an entire dominating set of G. This concept was studied in [13, 19]. A set X of elements in G is a total entire dominating set if every element in G is either adjacent or incident to at least one element in X. The total entire domination number $\varepsilon_t(G)$ of G in the minimum cardinality of a total entire dominating set of G. This concept was studied by Kulli and Sigarkanti in [18].

****Corresponding author: V. R. Kulli****
Department of Mathematics, Gulbarga University, Gulbarga 585 106, India.

For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ of S is defined by $N(S) = \bigcup_{v \in s} N(v)$ for all $v \in S$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. Let S be the set of vertices and let $u \in S$. The private neighbor set of u with respect to S is the set $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. For any edge $e \in E$, the open neighborhood of e is $N(e)$ and the closed neighborhood of e is $N[e] = N(e) \cup \{e\}$. If $F \subseteq E$, then $N(F) = \bigcup_{v \in s} N(e)$ and $N[F] = N(F) \cup F$. If $F \subseteq E$ and $e_1 \in F$, then the private neighbor of e_1 with respect to F is the set $pn[e_1, F] = \{ e_2 : N[e_2] \cap F = \{ e_1\}\}$. The degree of an edge uv is defined by deg u + deg v – 2. An edge uv is called an isolated edge if deg uv = 0. Let $\Delta'(G)$ denote the maximum degree among the edges of G.

In the cycle $C_9 = \{e_1, e_2, \ldots, e_9\}$, $F_1 = \{e_1, e_4, e_7\}$ and $F_2 = \{e_1, e_4, e_6, e_8\}$ are edge dominating sets of C_9. The induced subgraph $\langle N(F_1)\rangle$ has no isolated edges and the induced subgraph $\langle N(F_2)\rangle$ has isolated edges. Motivated by this example in [14] Kulli introduced the concept of neighborhood total edge domination number. In this paper, we study this parameter.

2. RESULTS

We assume throughout that G is a graph without isolated vertices and without isolated edges.

Definition 1: An edge dominating set F of a graph G is called a neighborhood total edge dominating set if the induced subgraph $\langle N(F)\rangle$ contains no isolated edges. The neighborhood total edge domination number $\gamma'_{nt}(G)$ of G is the minimum cardinality of a neighborhood total edge dominating set of G.

Definition 2: A neighborhood total edge dominating set is minimal if no proper subset of F is a neighborhood total edge dominating set.

Example 3: Consider the graph G as shown in Figure 1,

G:

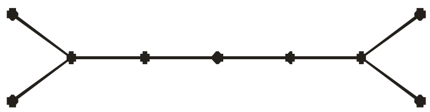

Figure-1

We see that $\gamma'(G) = 2$, $\gamma'_c(G) = 4$, $\gamma'_t(G) = 4$, $\gamma'_{nt}(G) = 2$.

Proposition 4: For a graph G,

$$\gamma'(G) \leq \gamma'_{nt}(G). \tag{1}$$

Proof: Every neighborhood total edge dominating set is an edge dominating set. Thus (1) holds. The graph G of Figure 1 achieves this bound.

Theorem 5: If P_p is a path with $p \geq 4$ vertices, then

$$\gamma'_{nt}(P_p) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Proof: Let $P_p = (v_1, v_2, \ldots, v_p)$ be a path with $p \geq 4$ vertices. If $p = r$ (mod 4), r = 0, 1 or 3, then $F=\{e_i: i=4k-2, 4k-1, k=1, 2, \ldots\}$ is a neighborhood total edge dominating set of P_p. If p=2 (mod 4), then $F \cup \{e_{p-2}\}$ is a neighborhood total edge dominating set of P_p. Thus

$$\gamma'_{nt}(P_p) \leq \left\lfloor \frac{p}{2} \right\rfloor.$$

If $p = r$ (mod 4), r = 0, 1 or 3, then $\gamma'_{nt}(P_p) \geq \gamma'_t(P_p) = \left\lfloor \frac{p}{2} \right\rfloor$. Further if p = 2 (mod 4), then for any γ'_t-set F of P_p, $\langle N(F)\rangle$ has at least one isolated edge. Thus $\gamma'_{nt}(P_p) \geq \left\lceil \frac{p}{2} \right\rceil \geq \left\lfloor \frac{p}{2} \right\rfloor$. Hence the result follows.

Corollary 6:. If P_p is a path with $p \geq 4$ vertices, then $\gamma'_{nt}(P_p) = \gamma'_t(P_p)$ if and only if p is even or p = 1 (mod 4).

Proof: Since $\gamma'_t(P_p) = \frac{p}{2}$, if p is even,

$= \left\lfloor \frac{p}{2} \right\rfloor$, if p = 1(mod 4), the result follows.

Theorem 7: If C_p is a cycle with $p \geq 3$ vertices, then

$$\gamma'_{nt}(C_p) = \left\lceil \frac{p}{3} \right\rceil + 1, \text{ if } p = 2 \text{ (mod3)},$$
$$= \left\lceil \frac{p}{3} \right\rceil, \text{ otherwise.}$$

Proof: Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$ be a cycle with $p \geq 3$ vertices. If $p = r$ (mod 3), r = 0 or 1, then $F = \{e_i : i = 3k - 2, k = 1, 2, \ldots\}$ is a neighborhood total edge dominating set of C_p. If p = 2(mod 3), then $F \cup \{e_p\}$ is a neighborhood total edge dominating set of C_p. Thus

$$\gamma'_{nt}(C_p) \leq \begin{cases} \left\lceil \frac{p}{3} \right\rceil + 1, & \text{if } p = 2(\text{mod } 3), \\ \left\lceil \frac{p}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

We have $\gamma'_{nt}(C_p) \geq \gamma'(C_p) = \left\lceil \frac{p}{3} \right\rceil$. If p = 2 (mod 3), then for any γ'-set of F of C_p, $\langle N(F) \rangle$ has at least one isolated edge. Thus $\gamma'_{nt}(C_p) \geq \left\lceil \frac{p}{3} \right\rceil + 1$. Hence the result follows.

Corollary 8: If C_p is a cycle with $p \geq 3$ vertices, then

$\gamma'_{nt}(C_p) = \gamma'_t(C_p)$ if and only if p = 4, 5 or 8,
$\gamma'_{nt}(C_p) = \gamma'_c(C_p)$ if and only if p = 3, 4 or 5.
$\gamma'_{nt}(C_p) = \gamma'(C_p)$ if and only if p = 0(mod 3) or p = 1(mod 3).

Proof: Since
$$\gamma'_t(C_p) = \frac{p}{2}, \text{ if } p = 0 \text{ (mod 4)}$$
$$= \left\lfloor \frac{p}{2} \right\rfloor, \text{ if } p = 1(\text{mod } 4) \text{ or } p = 3(\text{mod } 4)$$
$$= \left\lfloor \frac{p}{2} \right\rfloor + 1, \text{ if } p = 2(\text{mod } 4),$$
$$\gamma'_c(C_p) = p - 2,$$
$$\gamma'(C_p) = \left\lceil \frac{p}{3} \right\rceil,$$
the result follows.

Theorem 9: If $K_{m, n}$ is a complete bipartite graph with $2 \leq m \leq n$, then

$$\gamma'_{nt}(K_{m, n}) = m.$$

Proof: In $K_{m, n}$, v is a vertex such that deg $v = m$. Let F be the set of all edges incident with a vertex v. It is easy to see that F is an edge dominating set and the induced subgraph $\langle N(F) \rangle$ is connected and does not contain an isolated edge. Hence F is a neighborhood total edge dominating set.

Thus $\gamma'_{nt}(K_{m, n}) \leq |F|$ = deg $v = m$. Since $\gamma'(K_{m, n}) = m$, the theorem follows.

Theorem 10: If K_p is a complete graph with $p \geq 3$ vertices, then

$$\gamma'_{nt}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Proof: Let F be a maximum matching of K_p. Clearly F is an edge dominating set and also $\langle N(F) \rangle$ is connected and does not contain an isolated edge. Hence F is a neighborhood total edge dominating set. Thus

$$\gamma'_{nt}(K_p) \leq |F| = \left\lfloor \frac{p}{2} \right\rfloor.$$

Since $\gamma'(K_p) = \left\lfloor \frac{p}{2} \right\rfloor$, the result follows.

Theorem 11: A superset of a neighborhood total edge dominating set is a neighborhood total edge dominating set.

Proof: Let F be a neighborhood total edge dominating set of a graph G. Let $F_1 = F \cup \{e\}$, where $e \in E - F$. Then $e \in N(F_1)$ and F_1 is an edge dominating set of G. Suppose the induced subgraph $\langle N(F_1)\rangle$ contains an isolated edge e_1. Then $N(e_1) \subseteq F - N(F)$. Thus e_1 is an isolated edge in $\langle N(F)\rangle$, which is a contradiction. Thus $\langle N(F_1)\rangle$ has no isolated vertices. Therefore F_1 is a neighborhood total edge dominating set.

We establish a characterization of minimal neighborhood total edge dominating sets.

Theorem 12: A neighborhood total edge dominating set F of a graph G is minimal if and only if for every $e \in$ F, one of the following holds.

(i) $pn\,[e, F] \neq \phi$

(ii) there exists an edge $e_1 \in N(F - \{e\})$ such that $N(e_1) \cap N(F - \{e\}) = \phi$.

Proof: Let F be a minimal neighborhood total edge dominating set of G. Let $e \in F$. Then either $F - \{e\}$ is not an edge dominating set G or $F - \{e\}$ is an edge dominating set and the induced subgraph $\langle N(F - \{e\})\rangle$ contains an isolated vertex. Suppose $F - \{e\}$ is not an edge dominating set. Then $pn\,[e, F] \neq \phi$. Suppose $F - \{e\}$ is an edge dominating set and $e_1 \in N(F - \{e\})$ is an isolated edge in $\langle N(F - \{e\})\rangle$. Then $N(e_1) \cap N(F - \{e\}) = \phi$.

Conversely suppose F is a neighborhood total edge dominating set of G satisfying the conditions (i) and (ii). Then F is a minimal neighborhood total edge dominating set. Thus by Theorem 11, the result follows.

Theorem 13: Let T be a tree. Then $\gamma'_{nt}(T) = 1$ if and only if $T = K_{1,p}$, $p \geq 3$ or $S_{m,n}$, $2 \leq m \leq n$.

Proof: If $T = P_3$ or P_4, then clearly $\gamma'_{nt}(T) = 2$. Thus $T \neq P_3$ or P_4. Let $\gamma'_{nt}(T) = 1$. Let $F = \{e\}$ be the γ'_{nt}-set of T. Let $e = uv$. Since $T \neq P_3$, $\deg v \geq 3$. Suppose $\deg u = 2$. Then $\langle N(F)\rangle$ has two components in which one component is an isolated edge, which is a contradiction. This implies that $\deg u = 1$ or $\deg u \geq 3$. If $\deg u = 1$, then $\gamma'_{nt}(T) = 1$ and $T = K_{1,p}$, $p \geq 3$. If $\deg u \geq 3$, $\gamma'_{nt}(T) = 1$ and $T = S_{m,n}$, $2 \leq m \leq n$.

Converse is obvious.

Proposition 14: If $T = S_{1,p}$, $p \geq 0$, then $\gamma'_{nt}(T) = 2$.

Theorem 15: If G is a connected graph with $\Delta' < q - 1$, then $\gamma'_{nt}(G) \leq q - \Delta'$.

Proof: Let e be an edge of a connected graph G and $\deg e = \Delta'$. Since $\Delta' < q - 1$, there exist two adjacent edges e_1 and e_2 such that $e_1 \neq e_2$, $e_1 \in N(e)$ and $e_2 \notin N(e)$. Let $F = (N(e) - e_1) \cup \{e_2\}$. Then $|F| = \Delta'$. Further it is easy to see that $E - F$ is a neighborhood total edge dominating set of G. Thus $\gamma'_{nt}(G) \leq |E - F| = q - \Delta'$.

Theorem 16: For any graph G, $\gamma'_{nt}(G) = q$ if and only if $G = mP_3$.

Proof: Suppose $\gamma'_{nt}(G) = q$. On the contrary, assume $G \neq mP_3$. Then G has at least one component G_1 which is not P_3.

Clearly all edges of G_1 are not in a neighborhood total edge dominating set. Hence $\gamma'_{nt}(G) \neq q$, which is a contradiction. Hence $G = mP_3$.

Converse is obvious.

3. SOME OPEN PROBLEMS

The following are some problems for further investigation

Problem 1: Characterize graphs G for which $\gamma'_{nt}(G) = 1$.

Problem 2: Characterize graphs G for which $\gamma'_{nt}(G) = 2$.

Problem 3: Characterize trees T for which $\gamma'_{nt}(T) = 2$.

Problem 4: Characterize graphs G for which $\gamma'_{nt}(G) = q - \Delta'$.

Problem 5: Characterize trees T for which $\gamma'_{nt}(T) = q - \Delta'$.

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Source of Support: Nil, Conflict of interest: None Declared