



ZERO-FREE REGION FOR POLYNOMIALS WITH RESTRICTED REAL COEFFICIENTS

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ABSTRACT

In this paper we prove some extension of the Eneström-Kakeya theorem says that. Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ then all the zeros of $P(z)$ lie in $|z| \leq 1$. By relaxing the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.

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1. INTRODUCTION

The well known Results Eneström-Kakeya theorem [1, 2] in theory of the distribution of zeros of polynomials is the following.

Theorem: (A₁) Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ then all the zeros of $P(z)$ lie in $|z| \leq 1$.

Applying the above result to the polynomial $z^n P(\frac{1}{z})$ we get the following result:

Theorem: (A₂) If $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree n such that $0 < a_n \leq a_{n-1} \leq a_{n-2} \leq \dots \leq a_0$ then $P(z)$ does not vanish in $|z| < 1$

In the literature [3-8], there exist several extensions and generalizations of the Eneström-Kakeya Theorem.

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

Theorem: 1 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, k \geq 1$ and $ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and $(n-m)$ are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then

(i) all the zeros of $P(z)$ does not vanish in the disk $|z| < \frac{|a_0|}{k(|a_0|+a_0) + |a_n| - (|a_0|+a_n) + 2\rho + S_1}$ if both n and $(n-m)$ are even or odd

where $S_1 = 2[(a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

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(ii) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{k(|a_0|+a_0)+|a_n|-(|a_0|+a_n)+2\rho+S_2}$ if n is even and (n-m) is odd
(or) if n is odd and (n-m) is even

where $S_2 = 2[(a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m-2})]$

Corollary: 1 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, k \geq 1$ and

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(2k-1)a_0+2\rho+S_1}$ if both n and (n-m) are even or odd

where $S_1 = 2[(a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

(ii) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(2k-1)a_0+2\rho+S_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[(a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m-2})]$

Remark: 1 By taking $a_i > 0$ for $i = 0, 1, 2, \dots, n$, in theorem 1, then it reduces to Corollary 1.

Theorem: 2 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, 0 < r \leq 1$ and

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_n|+|a_0|-a_n-r(a_0+|a_0|)+2\rho+T_1}$ if both n and (n-m) are even or odd

where $T_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2})]$

(ii) all the zeros of P(z) does not vanish in the disk $||z| < \frac{|a_0|}{|a_n|+|a_0|-a_n-r(a_0+|a_0|)+2\rho+T_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2[(a_1 + a_3 + \dots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1})]$.

Corollary: 2 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, 0 < r \leq 1$ and

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \dots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(1-2r)a_0+2\rho+T_1}$ if both n and (n-m) are even or odd

where $T_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2})]$

(ii) all the zeros of P(z) does not vanish in the disk $||z| < \frac{a_0}{(1-2r)a_0+2\rho+T_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2[(a_1 + a_3 + \dots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1})]$.

Remark: 2 By taking $a_i > 0$ for $i = 0, 1, 2, \dots, n$, in theorem 2, then it reduces to Corollary 4.

Theorem: 3 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, k \geq 1$ and

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and $(n-m)$ are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then

(i) all the zeros of $P(z)$ does not vanish in the disk $|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$ if both n and $(n-m)$ are even or odd

where $U_1 = 2[(a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

(ii) all the zeros of $P(z)$ does not vanish in the disk $|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even

where $U_2 = 2[(a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m})]$

Corollary: 3 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, k \geq 1$ and

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and $(n-m)$ are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \dots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then

(i) all the zeros of $P(z)$ does not vanish in the disk $|z + k - 1| < \frac{a_0}{ka_0 + 2a_n + 2\rho + U_2}$ if both n and $(n-m)$ are even or odd

where $U_1 = 2[(a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

(ii) all the zeros of $P(z)$ does not vanish in the disk $|z + k - 1| < \frac{a_0}{ka_0 + 2a_n + 2\rho + U_2}$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even

where $U_2 = 2[(a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m})]$.

Remark: 3 By taking $a_i > 0$ for $i = 0, 1, 2, \dots, n$ in theorem 3, then it reduces to Corollary 3.

Theorem: 4 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, 0 < r \leq 1$ and $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and $(n-m)$ are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then

(i) all the zeros of $P(z)$ does not vanish in the disk $|z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1}$ if both n and $(n-m)$ are even or odd

where $V_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m})]$

(ii) all the zeros of $P(z)$ does not vanish in the disk $||z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_2}$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even

where $V_2 = 2[(a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1})]$.

Corollary: 4 Let $P(z) = \sum_{i=0}^n a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, 0 < r \leq 1$ and

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and $(n-m)$ are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \dots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \dots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{2a_n+(1-2r)a_0+V_1}$ if both n and (n-m) are even or odd where $V_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m})]$

(ii) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{2a_n+(1-2r)a_0+V_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[(a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1})]$.

Remark: 4 By taking $a_i > 0$ for $i = 0, 1, 2, \dots, n$, in theorem 4, then it reduces to Corollary 4.

2. PROOFS OF THE THEOREMS

Proof of the Theorem 1: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree n.

Let us consider the polynomial $J(z) = z^n P\left(\frac{1}{z}\right)$ and $R(z) = (z - 1)J(z)$ so that

$$\begin{aligned} R(z) &= (z - 1)(a_0 z^n + a_1 z^{n-1} + \dots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \dots + a_{n-1} z + a_n) \\ &= a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \dots + (a_{m-1} - a_m)z^{n-m+1} \\ &\quad + (a_m - a_{m+1})z^{n-m} + \dots + (a_{n-1} - a_n)z + a_n\} \end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, 2, \dots, n - 1$.

Now

$$|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \dots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \dots + |a_{n-1} - a_n||z| + |a_n| \}$$

$$\geq |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ |a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \dots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \dots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}]$$

$$\geq |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ k|a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \dots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \dots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |a_n| \}]$$

$$\geq |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ (k|a_0 - a_1|) + (k-1)|a_0| + (a_2 - a_1) + (a_2 - a_3) + \dots + (a_{n-m} - a_{n-m-1}) + (a_{n-m} - a_{n-m+1}) + \dots + (a_{n-3} - a_{n-2}) + (a_{n-2} - a_{n-1}) + (a_{n-1} - a_n) + |a_n| \}]$$

$$= |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1 \}]$$

where $S_1 = 2[(a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

$$\Rightarrow R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \{ k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1 \}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \{ k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1 \}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|} \{ k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1 \}$$

Since $P(z) = z^n J\left(\frac{1}{z}\right)$ it followed by replacing z by $\frac{1}{z}$, all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1}$$

if both n and (n-m) are even or odd

$$\text{where } S_1 = 2[(a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

$$\text{where } S_2 = 2[(a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m-2})]$$

This completes the proof of the Theorem 1.

Proof of the Theorem 2:

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n P\left(\frac{1}{z}\right)$ and $R(z) = (z-1)J(z)$ so that

$$\begin{aligned} R(z) &= (z-1)(a_0 z^n + a_1 z^{n-1} + \dots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \dots + a_{n-1} z + a_n) \\ &= a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \dots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} \\ &\quad + \dots + (a_{n-1} - a_n)z + a_n\} \end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, 2, \dots, n-1$.

$$\begin{aligned} \text{Now } |R(z)| &\geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \dots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} \\ &\quad + \dots + |a_{n-1} - a_n||z| + |a_n| \} \end{aligned}$$

$$\begin{aligned} &\geq |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ |a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \dots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} \\ &\quad + \dots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}] \end{aligned}$$

$$\begin{aligned} &\geq |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ r a_0 - a_1 - r a_0 + a_0 + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \dots + \\ &\quad |a_{m-1} - a_m| + |a_m - a_{m+1}| + \dots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |a_n| \}] \end{aligned}$$

$$\begin{aligned} &\geq |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ (a_1 - r a_0) + (1-r)|a_0| + (a_1 - a_2) + (a_3 - a_2) \\ &\quad + \dots + (a_{n-m-1} - a_{n-m}) + (a_{n-m} - a_{n-m+1}) + \dots + (a_{n-3} - a_{n-2}) + (a_{n-2} - a_{n-1}) \\ &\quad + (a_{n-1} + \rho - a_n) + \rho + |a_n| \}] \end{aligned}$$

if both n and (n-m) are even or odd

$$= |a_0||z|^n [|z| - \frac{1}{|a_0|} \{ |a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1 \}]$$

$$\text{Where } T_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2})]$$

$$\Rightarrow R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \{ |a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1 \}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \{ |a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1 \}$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in

$$|z| \leq \frac{1}{|a_0|} \{ |a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1 \}$$

Since $P(z) = z^n J(\frac{1}{z})$ it followed by replacing z by $\frac{1}{z}$, all the zeros of $P(z)$ lie in

$$|z| \geq \frac{|a_0|}{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1},$$

if both n and $(n-m)$ are even or odd.

Hence all the zeros $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1}$$

if both n and $(n-m)$ are even or odd

where $T_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2})]$

Similarly we can also prove for if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros $P(z)$ does not vanish in the disk.

$$|z| < \frac{|a_0|}{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_2}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even

where $T_2 = 2[(a_1 + a_3 + \dots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1})]$.

This completes the proof of the Theorem 2.

Proof of the Theorem 3: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n P(\frac{1}{z})$ and $R(z) = (z-1)J(z)$ so that

$$\begin{aligned} R(z) &= (z-1)(a_0 z^n + a_1 z^{n-1} + \dots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \dots + a_{n-1} z + a_n) \\ &= a_0 z^n (z+k-1) - \{ (ka_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \dots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} \\ &\quad + \dots + (a_{n-1} - a_n) z + a_n \} \end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < \text{for } i = 0, 1, 2, \dots, n-1$.

Now

$$\begin{aligned} |R(z)| &\geq |a_0| |z|^n |z+k-1| - \{ |ka_0 - a_1| |z|^n + |a_1 - a_2| |z|^{n-1} + \dots + |a_{m-1} - a_m| |z|^{n-m+1} \\ &\quad + |a_m - a_{m+1}| |z|^{n-m} + \dots + |a_{n-1} - a_n| |z| + |a_n| \} \\ &\geq |a_0| |z|^n |z|^n [|z+k-1| - \frac{1}{|a_0|} \{ |ka_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} \\ &\quad + \dots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \dots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}] \\ &\geq |a_0| |z|^n [|z+k-1| - \frac{1}{|a_0|} \{ |ka_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| \\ &\quad + \dots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \dots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - \rho - a_n + \rho| + |a_n| \}] \\ &\geq |a_0| |z|^n |z|^n [|z+k-1| - \frac{1}{|a_0|} \{ (ka_0 - a_1) + (a_2 - a_1) + (a_2 - a_3) \\ &\quad + \dots + (a_{n-m} - a_{n-m-1}) + (a_{n-m+1} - a_{n-m}) + \dots + (a_{n-2} - a_{n-3}) + (a_{n-1} - a_{n-2}) \\ &\quad + (a_n + \rho - a_{n-1}) + \rho + |a_n| \}] \end{aligned}$$

if both n and $(n-m)$ are even or odd

$$= |a_0| |z|^n [|z+k-1| - \frac{1}{|a_0|} \{ ka_0 + |a_n| + a_n + 2\rho + U_1 \}]$$

where $U_1 = 2[(a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

$$\Rightarrow R(z) > 0 \text{ if } |z + k - 1| > \frac{1}{|a_0|} \{ka_0 + |a_n| + a_n + 2\rho + U_1\}$$

This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk

$$|z + k - 1| \leq \frac{1}{|a_0|} \{ka_0 + |a_n| + a_n + 2\rho + U_1\}$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in

$$|z + k - 1| \leq \frac{1}{|a_0|} \{ka_0 + |a_n| + a_n + 2\rho + U_1\}$$

Since $P(z) = z^n J(\frac{1}{z})$ it followed by replacing z by $\frac{1}{z}$,

all the zeros of $P(z)$ lie in

$$|z + k - 1| \geq \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_1},$$

if both n and $(n-m)$ are even or odd.

Hence all the zeros $P(z)$ does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_1}$$

if both n and $(n-m)$ are even or odd

where $U_1 = 2[(a_2 + a_4 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1})]$

Similarly we can also prove for if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros $P(z)$ does not vanish in the disk.

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even

where $U_2 = 2[(a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m})]$.

This completes the proof of the Theorem 3.

Proof of the Theorem 4: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n P(\frac{1}{z})$

and $R(z) = (z - 1)J(z)$ so that

$$\begin{aligned} R(z) &= (z - 1)(a_0 z^n + a_1 z^{n-1} + \dots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \dots + a_{n-1} z + a_n) \\ &= a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \dots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} \\ &\quad + \dots + (a_{n-1} - a_n)z + a_n\} \end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < \text{for } i = 0, 1, 2, \dots, n - 1.$

Now

$$|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \dots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} \\ + \dots + |a_{n-1} - a_n||z| + |a_n| \}$$

$$\begin{aligned} &\geq |a_0||z|^n|z|^n \left[|z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \dots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} \right. \right. \\ &\quad \left. \left. + \dots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right\} \right] \\ &\geq |a_0||z|^n|z|^n \left[|z| - \frac{1}{|a_0|} \{ r a_0 - a_1 - r a_0 - a_0 + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| \right. \\ &\quad \left. + \dots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \dots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |a_n| \} \right] \\ &\geq |a_0||z|^n|z|^n \left[|z| - \frac{1}{|a_0|} \{ (a_1 - r a_0) + (1 - r)|a_0| + (a_1 - a_2) + (a_3 - a_2) \right. \\ &\quad \left. + \dots + (a_{n-m-1} - a_{n-m}) + (a_{n-m+1} - a_{n-m}) + \dots + (a_{n-2} - a_{n-3}) + (a_{n-1} - a_{n-2}) \right. \\ &\quad \left. + (a_n - a_{n-1}) + |a_n| \} \right] \end{aligned}$$

if both n and (n-m) are even or odd

$$\begin{aligned} &= |a_0||z|^n \left[|z| - \frac{1}{|a_0|} \{ |a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1 \} \right] \\ &\text{Where } V_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m})] \\ &\Rightarrow R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \{ |a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1 \} \end{aligned}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \{ |a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1 \}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|} \{ |a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1 \}$$

Since $P(z) = z^n J(\frac{1}{z})$ it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1}$$

if both n and (n-m) are even or odd

$$\text{where } V_1 = 2[(a_1 + a_3 + \dots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \dots + a_{n-m-2} + a_{n-m})]$$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

$$\text{where } V_2 = 2[(a_1 + a_3 + \dots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \dots + a_{n-m-3} + a_{n-m-1})].$$

This completes the proof of the Theorem 4.

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