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2-KNOT SYMMETRIC ALGEBRAS

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ABSTRACT

In this paper, we introduce a new class of algebras K_n, which we call 2-knot symmetric algebras. The reason for this name is that these new algebras have a basis consisting of knot diagrams. The multiplication of two of these graphs turns K_n into an associative algebra. By making use of conditional expectation and proving the non-degeneracy of the trace, we and also prove the semi simplicity of these algebras over $K_n(x)$.

Keywords: *Multiplication in K_n, Brauer algebra, knot graphs, semi simple.*

INTRODUCTION

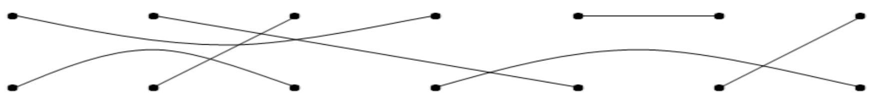
Brauer [1] introduced certain algebras, known as Brauer's algebras, in connection with the problem of the decomposition of a tensor product representation into irreducible. These algebras have a basis consisting of undirected graphs. Wenzl[2] obtained the structure of Bauer algebras D_{n+1} by making use of conditional expectations and by an inductive procedure from the structures of D_{n-1} and D_n. Parvathi and Kamaraj [3] introduced signed Brauer's algebras, which have a basis consisting of signed diagrams. Kamaraj and Mangayarkarasi [4] introduced knot diagrams using Brauer graphs without horizontal edges. They used only two types of knot. We are motivated by the above to introduce a new multiplication among the generators of 2-knot multiplication. We call these 2-knot symmetric algebras, and we also prove the semisimplicity of K_n.

1. PRELIMINARIES

We state the basic definitions and some known results that will be used in this paper.

1.1 Brauer algebras

Definition [1] A Brauer graph is a graph on 2n vertices with n edges, the vertices being arranged in two rows and each row consisting of n vertices, and every vertex is the vertex of only one edge.



1.2 Definition [1], [2]

Define the Brauer Algebra D_n over the field of rational functions $C(x)$ as follows.

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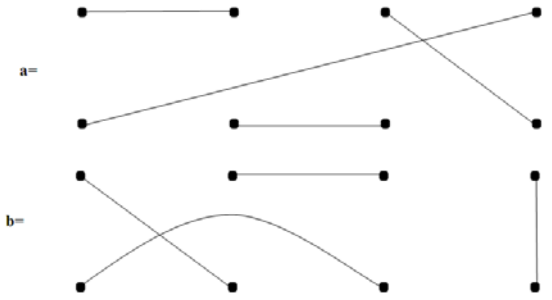
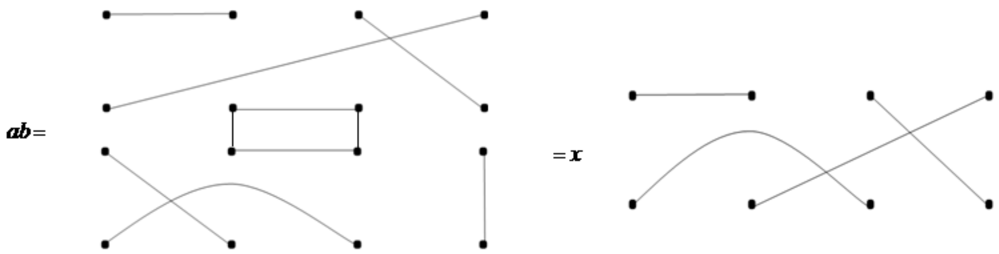
For n = 0 let $D_0 = C(x)$. For $n > 0$, a linear basis of the $C(x)$ algebra D_n is given by the graphs with n edges and 2n vertices, arranged in two lines of n vertices each. In these graphs each edge belongs to exactly two vertices and each vertex belongs to exactly one edge. Two examples for graphs in D_4 are



It is easy to see that we have $2n-1$ possibilities for joining the first vertex with another one, then $2n-3$ possibilities for joining the next one, and so on. So the dimension of D_n is $(2n-1).(2n-3)$…3.1. To define multiplication in D_n, it is enough to define the product a b for two graphs a and b. This is done in a similar way as for braids, by the following rules.

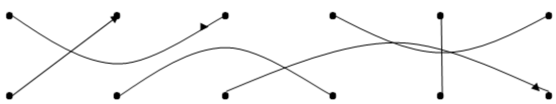
1. Draw b below a.
2. Connect the i-th upper vertex of b with the i-th lower vertex of a.
3. Let d be the number of cycles in the graph obtained in 2, and let c be this graph without the cycles. Then we define $a.b = x^d.c$

Example:

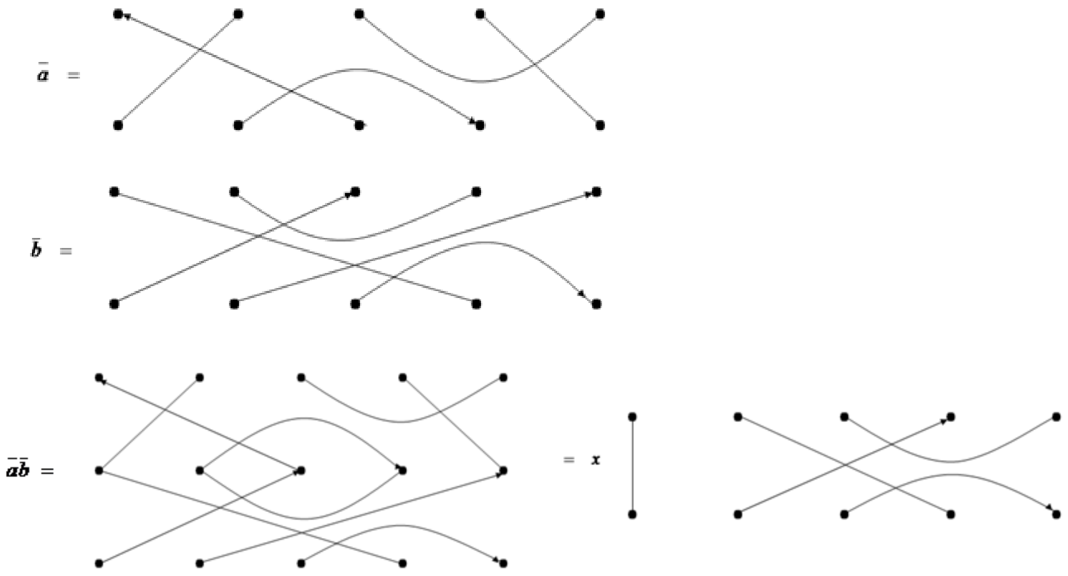


1.3 Signed Brauer algebras [3]

A signed diagram is a Brauer graph in which every edge is labeled by a + or – sign.



1.4 Definition [3]:Let $\overline{V_n}$ denote the set of all singed Brauer graphs on 2n vertices with n singed edges.Let $\overline{D_n}(x)$ denote the linear span of $\overline{V_n}$ where X is an indeterminate. The dimension of $\overline{D_n}(x)$ is 2^n (2n)! $=2^n$ (2n-1)(2n – 3)…3.1.Let $\overline{a}, \overline{b} \in \overline{V_n}$ .Since a, b are Brauer graphs, $ab = x^d c$, the only thing we have to do is to assign a direction for every edge. An edge α in the product $\overline{ab}$ will be labeled as a + or a- sign according as the number of negative edges involved from $\overline{a}$ and $\overline{b}$ to make α is even or odd. A loop β is said to be a positive or a negative loop in $\overline{ab}$ according as the number of negative edges involved in the loop is even or odd. Then $\overline{ab} = x^{2d_1+d_2}$ where d_1 is the number of positive loops and d_2 is the number of negative loops. Then is a finite dimensional algebra.



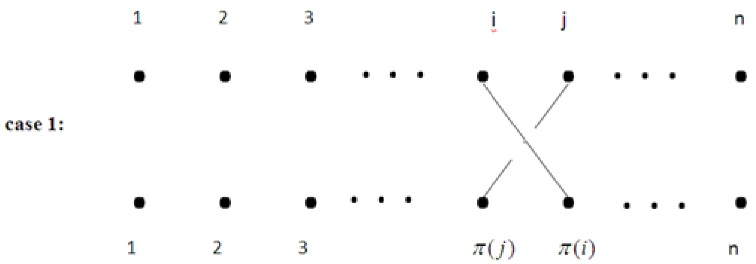
1.5 Knot Graphs [4]

Let S_n be the symmetric group of order n, and $\pi \in S_n$. A knot graph of order n is a special graph which is defined from π as follows.

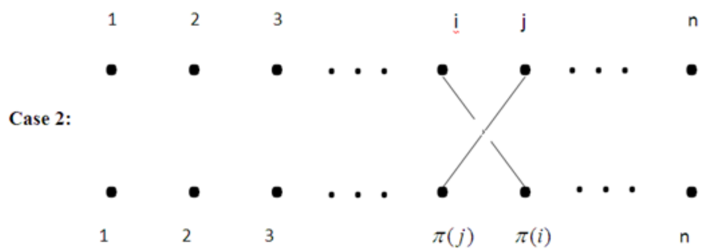
1.5 Definition [4]

Let $\pi \in S_n$, then π can be represented by a graph, which is called the Brauer diagram. Consider two edges (i ,π(i)) and (j, π(j)), where the vertices i and j are in the upper row and π(i) and π(j) are in the lower row.

If i<j and $\pi(i) < \pi(j)$, then edges are drawn in two cases as shown below.



In case 1, (i,$\pi(i)$) is the upper edge and (j, $\pi(j)$) is the lower edge. It can also be said that the edge (j, $\pi(j)$) is lower than the edge (i, $\pi(i)$).



In case 2, the edge (j, $\pi(j)$) is higher than (i , $\pi(i)$), or (i, $\pi(i)$) is lower than (j, $\pi(j)$). The above graph is called a knot graph of order n.

2. 2-KNOT SYMMETRIC ALGEBRAS

Notations: Let F denote a field and F(x) be the field of fractions, where x is indeterminate. Let S_n be the symmetric group of order n and let $\pi \in S_n$. Then π can be represented as a graph in which the vertices of π are represented in two rows such that each row contains n vertices. The vertices of each row are indexed by 1,2…n from left to right in order. Let $E(\pi)$ denote the set of all edges of π.
i.e. $E(\pi) = \{e_i = (i, \pi(i)); 1 \le i \le n\}$

Define $A_\pi = \{a_{ij} = (e_i, e_j); i < j\}$
$B_\pi = \{b_{ij} = a_{ij}; \quad \pi(i) < \pi(j)\}$

2.1 Remark

Let R_π denote the collection of all symmetric knot graphs of order n derived from π

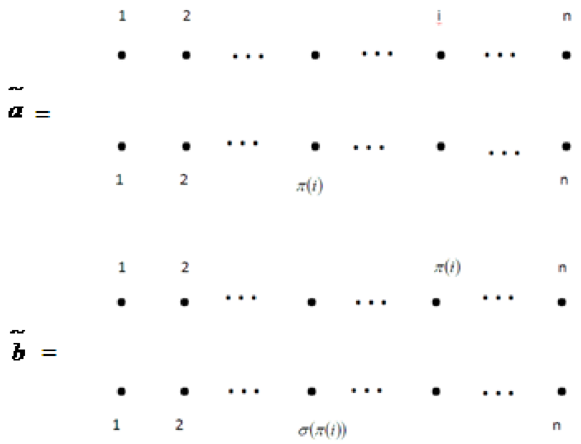
Let $K_n = \bigcup_{\pi \in S_n} R_\pi$

2.2 Remark: Let $\sigma, \pi \in S_n$. For the edge $\gamma_i = (i, \sigma.\pi(i)) \in E(\sigma \cdot \pi)$ there are edges $\alpha_i = (i, \pi(i)) \in E(\pi)$ and $\beta_i = (\pi(i), \sigma.\pi(i)) \in E(\sigma)$

2.3 Multiplication in K_n

We have introduced 2-knot multiplication among knot graphs in K_n. Now we define a product among the elements in K_n.

2.4 Definition: Let $\tilde{a}, \tilde{b}$ be elements in $K_n(x)$. The product of two diagrams $\tilde{a}$ and $\tilde{b}$ of n vertices is determined by putting the diagram $\tilde{a}$ at the top and $\tilde{b}$ below. The vertices of $\tilde{a}$ and $\tilde{b}$ will be as shown below:



Let $\tilde{a} \in K_\pi$ and $\tilde{b} \in K_\sigma$, then the product $\tilde{a}\tilde{b} \in K_{\sigma.\pi}$ is one of the cases mentioned below.

Case-1:

$\tilde{a}\tilde{b}(\gamma_i, \gamma_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i); \quad (\alpha_i, \alpha_j) \in B_\pi$

If α_i is higher (lower) than α_j, then $(\gamma_i, \gamma_j) \in B_{\sigma.\pi}$, where γ_i is higher (lower) than γ_j.

(i) If α_i is higher (lower) than α_j and β_i is higher (lower) than β_j, then $(\gamma_i, \gamma_j) \in B_{\sigma.\pi}$, where γ_i is lower (higher) than γ_j.

(ii) If α_i is lower (higher) than α_j and β_i is higher (lower) than β_j, then $(\gamma_i, \gamma_j) \notin B_{\sigma.\pi}$

Case-2:

$\tilde{a}\tilde{b}(\gamma_i, \gamma_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j); \quad (\alpha_i, \alpha_j) \notin B_\pi$

If β_i is higher (lower) than β_j, then $(\gamma_i, \gamma_j) \notin B_{\sigma.\pi}$, where γ_i is higher (lower) than γ_j.

2.5 Remark: Let $\sigma, \pi, \delta \in S_n$. For the edge $\eta_i = (i, \delta.(\sigma.\pi)(i)) \in E(\delta \cdot (\sigma.\pi))$, there are corresponding edges:

$\alpha_i = (i, \pi(i)) \in E(\pi), \quad \beta_i = (\pi(i), \sigma.\pi(i)) \in E(\sigma), \gamma_i = (\sigma \cdot \pi(i), \delta.(\sigma.\pi(i))) \in E(\delta)$

Let $\rho_i = (i, \sigma.\pi(i)) \in E(\sigma.\pi), \quad \xi_i = (\pi(i), \delta.\sigma.\pi(i)) \in E(\delta.\sigma)$

2.6 Theorem: If $\tilde{a}, \tilde{b}, and\ \tilde{c}$ are elements in $K_n(x)$, then $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$.

Proof: Let $\tilde{a} \in K_\pi, \tilde{b} \in K_\sigma\ and\ \tilde{c} \in K_\delta$, where $\pi, \sigma, \delta \in S_n$.

Claim: $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-1: Let $(\alpha_i, \alpha_j) \in B_\pi$, where α_i is higher than α_j, $(\beta_j, \beta_i) \in B_\sigma$, where β_j is higher than β_i and $(\gamma_i, \gamma_j) \in B_\delta$, where γ_i is higher than γ_j.

$\tilde{a}\tilde{b}(\rho_i, \rho_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i); \quad (\rho_i, \rho_j) \in B_{\sigma \bullet \pi}$, where ρ_i is lower than ρ_j.

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i, \eta_j) = \tilde{a}\tilde{b}(\rho_i, \rho_j) \bullet \tilde{c}(\gamma_i, \gamma_j)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\eta_i, \eta_j) \notin B_{\delta \bullet \sigma \bullet \pi}$$

$\tilde{b}\tilde{c}(\xi_j, \xi_i) = \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\xi_j, \xi_i) \in B_{\delta \bullet \sigma}$, where ξ_j is lower than ξ_i.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i, \eta_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}\tilde{c}(\xi_j, \xi_i)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\eta_i, \eta_j) \notin B_{\delta \bullet \sigma \bullet \pi}$$

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-2: Let $(\alpha_i, \alpha_j) \in B_\pi$, where α_i is higher than α_j, $(\beta_j, \beta_i) \in B_\sigma$, where β_j is higher than β_i and $(\gamma_i, \gamma_j) \notin B_\delta$.

$\tilde{a}\tilde{b}(\rho_i, \rho_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i); \quad (\rho_i, \rho_j) \in B_{\sigma \bullet \pi}$, where ρ_i is lower than ρ_j.

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i, \eta_j) = \tilde{a}\tilde{b}(\rho_i, \rho_j) \bullet \tilde{c}(\gamma_i, \gamma_j)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\eta_i, \eta_j) \in B_{\delta \bullet \sigma \bullet \pi}$$

where η_i is lower than η_j. $\tilde{b}\tilde{c}(\xi_j, \xi_i) = \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_i, \gamma_j)$; $(\xi_j, \xi_i) \in B_{\delta \bullet \sigma}$, where ξ_j is lower than ξ_i.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i, \eta_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}\tilde{c}(\xi_j, \xi_i)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\eta_i, \eta_j) \in B_{\delta \bullet \sigma \bullet \pi},$$

where η_i is lower than η_j.

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-3: Let $(\alpha_i, \alpha_j) \notin B_\pi$, $(\beta_i, \beta_j) \in B_\sigma$, where β_i is higher than β_j and $(\gamma_j, \gamma_i) \in B_\delta$, where γ_j is higher than γ_i.

$\tilde{a}\tilde{b}(\rho_i, \rho_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j)$; $(\rho_i, \rho_j) \in B_{\sigma \bullet \pi}$, where ρ_i is higher than ρ_j.

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i, \eta_j) = \tilde{a}\tilde{b}(\rho_i, \rho_j) \bullet \tilde{c}(\gamma_j, \gamma_i)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j) \bullet \tilde{c}(\gamma_j, \gamma_i); \quad (\eta_i, \eta_j) \in B_{\delta \bullet \sigma \bullet \pi}$$

where η_i is lower than η_j.

$\tilde{b}\tilde{c}(\xi_i, \xi_j) = \tilde{b}(\beta_i, \beta_j) \bullet \tilde{c}(\gamma_j, \gamma_i)$; $(\xi_i, \xi_j) \in B_{\delta \bullet \sigma}$, where ξ_i is lower than ξ_j.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i, \eta_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}\tilde{c}(\xi_i, \xi_j)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j) \bullet \tilde{c}(\gamma_j, \gamma_i); \quad (\eta_i, \eta_j) \in B_{\delta \bullet \sigma \bullet \pi},$$

where η_i is lower than η_j.

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-4: Let $(\alpha_i, \alpha_j) \in B_\pi$, where α_i is higher than α_j, $(\beta_j, \beta_i) \notin B_\sigma$, and $(\gamma_j, \gamma_i) \in B_\delta$, where γ_j is lower than γ_i.

$\tilde{a}\tilde{b}(\rho_i, \rho_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i)$; $(\rho_i, \rho_j) \in B_{\sigma \bullet \pi}$, where ρ_i is higher than ρ_j.

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i, \eta_j) = \tilde{a}\tilde{b}(\rho_i, \rho_j) \bullet \tilde{c}(\gamma_j, \gamma_i)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_j, \gamma_i); \quad (\eta_i, \eta_j) \notin B_{\delta \bullet \sigma \bullet \pi}$$

$\tilde{b}\tilde{c}(\xi_j, \xi_i) = \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_j, \gamma_i)$; $(\xi_j, \xi_i) \in B_{\delta \bullet \sigma}$, where ξ_j is lower than ξ_i.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i,\eta_j)=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}\tilde{c}(\xi_j,\xi_i)$$

$$=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_j,\beta_i)\bullet\tilde{c}(\gamma_j,\gamma_i);\quad (\eta_i,\eta_j)\notin B_{\delta\bullet\sigma\bullet\pi},$$

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c}=\tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-5: Let $(\alpha_i,\alpha_j)\notin B_\pi$, $(\beta_i,\beta_j)\notin B_\sigma$, and $(\gamma_i,\gamma_j)\notin B_\delta$.

$$\tilde{a}\tilde{b}(\rho_i,\rho_j)=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_i,\beta_j);\quad (\rho_i,\rho_j)\notin B_{\sigma\bullet\pi}$$

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i,\eta_j)=\tilde{a}\tilde{b}(\rho_i,\rho_j)\bullet\tilde{c}(\gamma_i,\gamma_j)$$

$$=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_i,\beta_j)\bullet\tilde{c}(\gamma_i,\gamma_j);\quad (\eta_i,\eta_j)\notin B_{\delta\bullet\sigma\bullet\pi}$$

$$\tilde{b}\tilde{c}(\xi_i,\xi_j)=\tilde{b}(\beta_i,\beta_j)\bullet\tilde{c}(\gamma_i,\gamma_j);\quad (\xi_i,\xi_j)\notin B_{\delta\bullet\sigma}$$

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i,\eta_j)=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}\tilde{c}(\xi_i,\xi_j)$$

$$=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_i,\beta_j)\bullet\tilde{c}(\gamma_i,\gamma_j);\quad (\eta_i,\eta_j)\notin B_{\delta\bullet\sigma\bullet\pi},$$

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c}=\tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-6: Let $(\alpha_i,\alpha_j)\notin B_\pi$, $(\beta_i,\beta_j)\in B_\sigma$, where β_i is higher than β_j and $(\gamma_j,\gamma_i)\notin B_\delta$.

$$\tilde{a}\tilde{b}(\rho_i,\rho_j)=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_i,\beta_j);\quad (\rho_i,\rho_j)\in B_{\sigma\bullet\pi}$$, where ρ_i is higher than ρ_j.

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i,\eta_j)=\tilde{a}\tilde{b}(\rho_i,\rho_j)\bullet\tilde{c}(\gamma_j,\gamma_i)$$

$$=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_i,\beta_j)\bullet\tilde{c}(\gamma_j,\gamma_i);\quad (\eta_i,\eta_j)\in B_{\delta\bullet\sigma\bullet\pi}$$

where η_i is higher than η_j.

$$\tilde{b}\tilde{c}(\xi_i,\xi_j)=\tilde{b}(\beta_i,\beta_j)\bullet\tilde{c}(\gamma_j,\gamma_i);\quad (\xi_i,\xi_j)\in B_{\delta\bullet\sigma}$$, where ξ_i is higher than ξ_j.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i,\eta_j)=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}\tilde{c}(\xi_i,\xi_j)$$

$$=\tilde{a}(\alpha_i,\alpha_j)\bullet\tilde{b}(\beta_i,\beta_j)\bullet\tilde{c}(\gamma_j,\gamma_i);\quad (\eta_i,\eta_j)\in B_{\delta\bullet\sigma\bullet\pi},$$

where η_i is higher than η_j.

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c}=\tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-7: Let $(\alpha_i, \alpha_j) \in B_\pi$, where α_i is higher than α_j, $(\beta_j, \beta_i) \notin B_\sigma$, and $(\gamma_j, \gamma_i) \notin B_\delta$.

$\tilde{a}\tilde{b}(\rho_i, \rho_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i)$; $\quad (\rho_i, \rho_j) \in B_{\sigma\bullet\pi}$, where ρ_i is higher than ρ_j.

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i, \eta_j) = \tilde{a}\tilde{b}(\rho_i, \rho_j) \bullet \tilde{c}(\gamma_j, \gamma_i)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_j, \gamma_i); \quad (\eta_i, \eta_j) \in B_{\delta\bullet\sigma\bullet\pi}$$

where η_i is higher than η_j.

$\tilde{b}\tilde{c}(\xi_j, \xi_i) = \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_j, \gamma_i)$; $\quad (\xi_j, \xi_i) \in B_{\delta\bullet\sigma}$, where ξ_j is higher than ξ_i.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i, \eta_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}\tilde{c}(\xi_j, \xi_i)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_j, \beta_i) \bullet \tilde{c}(\gamma_j, \gamma_i); \quad (\eta_i, \eta_j) \in B_{\delta\bullet\sigma\bullet\pi},$$

where η_i is higher than η_j.

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$

Case-8: Let $(\alpha_i, \alpha_j) \notin B_\pi$, $(\beta_i, \beta_j) \notin B_\sigma$, and $(\gamma_i, \gamma_j) \in B_\delta$, where γ_i is higher than γ_j,

$\tilde{a}\tilde{b}(\rho_i, \rho_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j)$; $\quad (\rho_i, \rho_j) \notin B_{\sigma\bullet\pi}$

$$\left(\tilde{a}\tilde{b}\right)\tilde{c}(\eta_i, \eta_j) = \tilde{a}\tilde{b}(\rho_i, \rho_j) \bullet \tilde{c}(\gamma_i, \gamma_j)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\eta_i, \eta_j) \in B_{\delta\bullet\sigma\bullet\pi}$$

where η_i is higher than η_j.

$\tilde{b}\tilde{c}(\xi_i, \xi_j) = \tilde{b}(\beta_i, \beta_j) \bullet \tilde{c}(\gamma_i, \gamma_j)$; $\quad (\xi_i, \xi_j) \in B_{\delta\bullet\sigma}$, where ξ_i is higher than ξ_j.

$$\tilde{a}\left(\tilde{b}\tilde{c}\right)(\eta_i, \eta_j) = \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}\tilde{c}(\xi_i, \xi_j)$$

$$= \tilde{a}(\alpha_i, \alpha_j) \bullet \tilde{b}(\beta_i, \beta_j) \bullet \tilde{c}(\gamma_i, \gamma_j); \quad (\eta_i, \eta_j) \in B_{\delta\bullet\sigma\bullet\pi},$$

where η_i is higher than η_j.

Therefore $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$

Similarly for the lower edges in $\left(\tilde{a}\tilde{b}\right)\tilde{c} = \tilde{a}\left(\tilde{b}\tilde{c}\right)$,

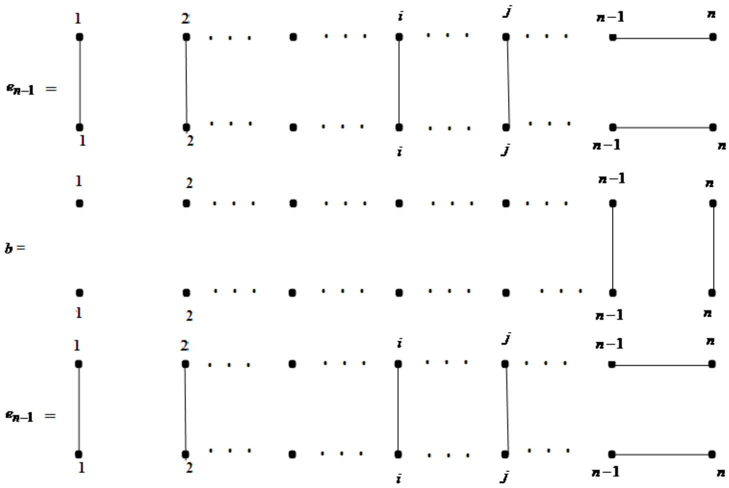
thus proving the associativity of the algebra.

2.7 Result: The free algebra generated by R_n over $F(x)$ is called a 2-knot symmetric algebra. It is denoted by K_n or $K_n(x)$.

3. SEMISIMPLICITY OF K_n

To define conditional expectation, we first prove the following cases. Let $e_{n-1} \in D_n, b \in K_\pi, \quad \pi \in S_{n-1}$

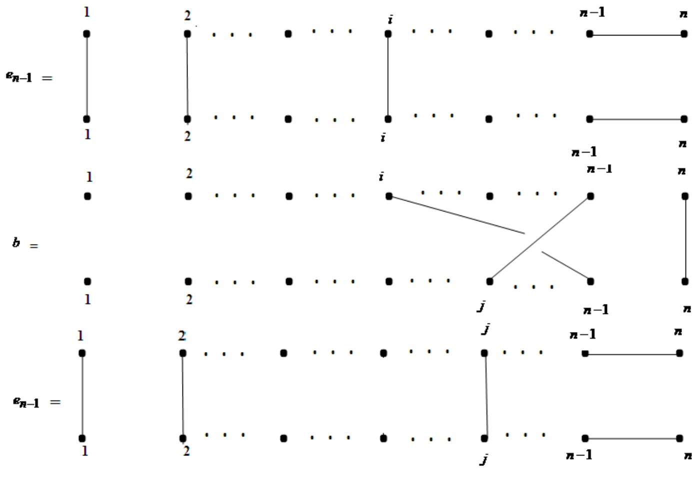
Case-1: Let $b \in K_{n-2}$ and $(\alpha_i, \alpha_{n-1}) \notin B_\pi$, where $\alpha_i = (i, n-1), \alpha_{n-1} = (n-1,\ j)$



The product of $e_{n-1}be_{n-1} \ = \ x^2\, be_{n-1}$ where $b' = x^2 b$

$$\left(\varepsilon_{n-1}(b)\right) = \frac{1}{x^2}\left(x^2 b\right) = b$$

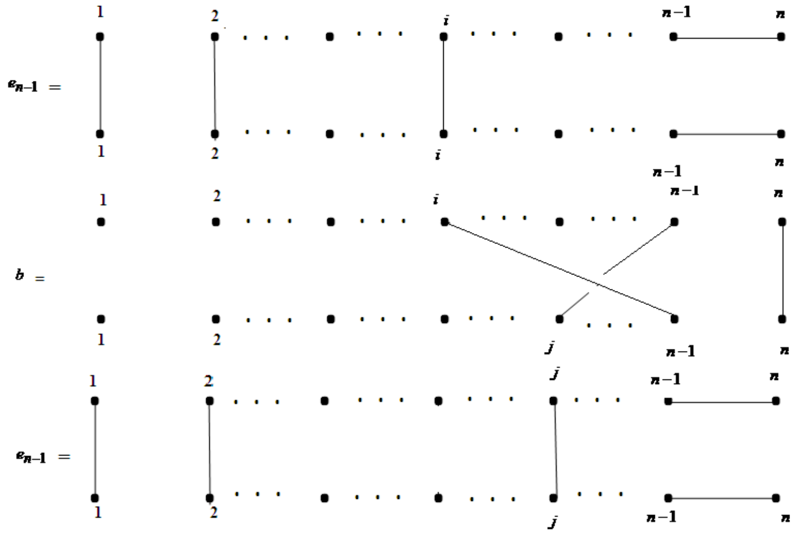
Case-2: Let $b \in K_{n-2}$ and $(\alpha_i, \alpha_{n-1}) \in B_\pi$, where $\alpha_i = (i, n-1), \alpha_{n-1} = (n-1,\ j)$ α_i is lower than α_j



The product of $e_{n-1}be_{n-1} \ = \ x^4\, b'' e_{n-1}$ where $b' = x^4 b''$

$$\left(\varepsilon_{n-1}(b)\right) = \frac{1}{x^2}\left(x^4 b''\right) = x^2 b''$$

Case-3: Let $b \in K_{n-2}$ and $(\alpha_i, \alpha_{n-1}) \in B_\pi$, where $\alpha_i = (i, n-1), \alpha_{n-1} = (n-1,\ j)$ α_i is higher than α_j



The product of $e_{n-1}be_{n-1} = be_{n-1}$ $\quad where\ b' = b$

$$\left(\varepsilon_{n-1}(b)\right) = \frac{1}{x^2}(b)$$

3.1 Definition: Define $\varepsilon_{n-1} : K_{n-1} \to K_{n-2}$ as follows: for every $b \in K_{\pi}$, there exists $b' \in K_{\sigma}, \sigma \in S_{n-2}$ such that $e_{n-1}be_{n-1} = b'e_{n-1}$. Now we define $\varepsilon_{n-1}(b) = \dfrac{b'}{x^2}$.

3.2 Definition: A trace $tr : K_{n-1} \to F(x)$ is defined inductively by:

(i). $tr(1) = 1$

(ii). $tr(b) = tr(\varepsilon_{n-1}(b)) = tr(\dfrac{b'}{x^2})$

3.3 Notation: $A_n = \{tr(b) : b \in K_n\}$

3.4 Example for trace of K_2

$tr : K_2 \to F(x)$

If b_1, b_2, b_3 are the generators of K_2 where $b_1, b_2, b_3 \in K_{\pi}$ & $\pi \in S_2$

Case-1: To compute $e_2b_1e_2$ where $e_2 \in K_2$ & $(\alpha_1, \alpha_2) \notin B_{\pi}$

$$e_2\ b_1\ e_2 = b'\ e_2, \quad where \quad b' = 1 \in S_1$$

$$\left(\varepsilon_{n-1}(b_1)\right) = \frac{1}{x^2}\left(x^2\right) = 1$$

$$tr(b_1) = tr(\varepsilon_{n-1}(b_1)) = tr(1) = 1$$

Case-2: To compute $e_2b_2e_2$ where $e_2 \in K_2$ & $(\alpha_1, \alpha_2) \in B_{\pi}$ and α_1 is lower than α_2

$$e_2b_2e_2 = x^4 b'' e_2, \ where\ b' = x^4 b''$$

$$\left(\varepsilon_{n-1}(b_2)\right) = \frac{1}{x^2}(x^4 b'') = x^2, where\, b'' = 1 \in S_1$$

$$tr(b_2) = tr(\varepsilon_{n-1}(b_2)) = tr(x^2) = x^2$$

Case-3: To compute $e_2b_2e_2$ where $e_2 \in K_2$ & $(\alpha_1, \alpha_2) \in B_\pi$ and α_1 is higher than α_2

$e_2b_3e_2 = b' e_2, where\ b' = 1 \in S_1$

$(\varepsilon_{n-1}(b_3)) = \frac{1}{x^2}$

$tr(b_3) = tr(\varepsilon_{n-1}(b_3)) = tr\left(\frac{1}{x^2}\right) = \frac{1}{x^2}$

Hence $A_2 = \{1, x^2, \frac{1}{x^2}\}$

3.5 Theorem: $tr : K_{n-1} \to F(x)$, if $\{ b_1, b_2, b_3, \ldots b_n \}$ are generators of K_{n-1}, then

$\{tr(b_1), tr(b_2), \ldots tr(b_n)\} = \{1, x^2, x^4 \ldots x^{2(n-1)}, \frac{1}{x^2}, \ldots \frac{1}{x^{2(n-1)}}\}$

Proof: Let us prove the theorem by induction on 'n'

For n = 2, that is $A_2 = \{1, x^2, \frac{1}{x^2}\}$

Hence the result is true for n=2.

Let us assume that the result is true for n=k.

Hence $A_k = \{tr(b) : b \in K_k\}$

That is $\{1, x^2, x^4 \ldots x^{2(k-1)}, \frac{1}{x^2}, \ldots \frac{1}{x^{2(k-1)}}\}$

For n =k+1

$A_{k+1} = \{tr(b) : b \in K_{k+1}\}$

Case-1: Let $b \in K_k$ and $(\alpha_i, \alpha_{k+1}) \notin B_\pi$, where $\pi \in S_n$

$e_{k+1}b = b' e_{k+1}, b' \in S_n$

$\varepsilon_{k+1} : K_{k+1} \to F(x)$

$(\varepsilon_{k+1}\ (b)) = \frac{1}{x^2}(x^2 b') = b'$

$tr(b) = tr(\varepsilon_{k+1}(b)) = tr(b')$ where $tr(b') \in A_k$

That is $tr(b) \in \{1, x^2, x^4 \ldots x^{2k}, \frac{1}{x^2}, \ldots \frac{1}{x^{2k}}\}$

Case-2: Let $b \in K_k$ and $(\alpha_i, \alpha_{n+1}) \notin B_\pi$, where $\pi \in S_n$ and α_i is lower than α_{k+1}

$e_{k+1}b = b' e_{k+1}, b' \in S_n$

$(\varepsilon_{k+1}\ (b)) = \frac{1}{x^2}(b)$

$tr(b) = tr(\varepsilon_{k+1}(b)) = tr(b') = tr(\frac{1}{x^2}b') = \frac{1}{x^2}tr(b) = \frac{1}{x^2}\{1, x^2, x^4 \ldots x^{2(k-1)}, \frac{1}{x^2}, \ldots \frac{1}{x^{2(k-1)}}\} =$

$\{1, x^2, x^4 \ldots x^{2k}, \frac{1}{x^2}, \ldots \frac{1}{x^{2k}}\}$

Case-3: Let $b \in K_k$ and $(\alpha_i, \alpha_{n+1}) \notin B_\pi$, where $\pi \in S_n$ and α_i is lower than α_{k+1}

$$e_{k+1}b = b' e_{k+1}, b' \in S_n$$

$$(\varepsilon_{k+1}(b)) = \frac{1}{x^2}(x^4 b'') = x^2 b'$$

$$tr(b) = tr(\varepsilon_{k+1}(b)) = tr(x^2 b') = x^2\, tr(b') = x^2 tr(b) = x^2\{1, x^2, x^4 \ldots x^{2(k-1)}, \frac{1}{x^2}, \ldots \frac{1}{x^{2(k-1)}}\} =$$

$$\{1, x^2, x^4 \ldots x^{2k}, \frac{1}{x^2}, \ldots \frac{1}{x^{2k}}\}$$

Hence the result is true for n= k+1.

By the induction hypothesis, the theorem is true for all n.

3.6. Theorem: $tr : K_n(x) \to F(x)$ is non-degenerate.

Let $X = \sum_i \lambda_i b_i, \{b_i\}$ be the basis of K_n; $\lambda_i \in F(x)$

$tr(XY) = 0$ for all $y \in K_n(x)$. In particular $tr(Xb_j) = 0$ *for all* j

$$tr\left(\sum_i \lambda_i b_i b_j\right) = 0$$

$$\left(\sum_i \lambda_i tr(b_i b_j)\right) = 0$$

Put $K = tr(b_i b_j)$ & $Q_f(x) = \det(K)$ is a non-zero polynomial.

Hence $\lambda_i = 0$ for all i, which implies X = 0.

3.7 Theorem: The generalized knot symmetric algebra $K_n(x)$ is semisimple.

Proof: Since the trace is non-degenerate, by the above theorem the algebra $K_n(x)$ is semisimple.

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