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ON THE BLOCK-EDGE TRANSFORMATION GRAPHS G^{ab}

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ABSTRACT

In this paper, we introduce block-edge transformation graphs. We investigate some basic properties such as connectedness, graph equations and diameters of the block-edge transformation graphs.

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Keywords: *line graph, jump graph, plick graph, block-edge transformation graphs G^{ab}.*

1. INTRODUCTION

All the graphs considered here are finite, undirected without loops or multiple edges. We refer to [4] for unexplained terminology and notation. A *block* of a graph is connected nontrivial graph having no cutvertices. Let $G = (V, E)$ be a graph with block set $U(G)$={B_i; B_i is a block of G}. If a block $B \in U(G)$ with the edge set $\{e_1, e_2, \ldots, e_r; r \geq 1\}$, then we say that the edge e_i and block B are incident with each other, where $1 \leq i \leq r$. The block B and an edge e are said to be adjacent if e is adjacent with at least one incident edge of B, otherwise not adjacent. The *line graph* $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *jump graph* $J(G)$ of a graph G is the graph whose the vertex set is the edge set of G and two vertices of $J(G)$ are adjacent if and only if the corresponding edges in G are not adjacent in G. The *plick graph* $P(G)$ of a graph G is the graph whose set of vertices is the union of the set of edges and blocks of G and in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or one is corresponds to an edge and other is corresponds to a block are incident. This concept is introduced by V. R. Kulli [6] and was studied in [2, 3, 7]. Inspired by the definition of plick graph of a graph, we define the following block-edge transformation graphs.

Definition: Let $G = (V, E)$ be a graph with a block set $U(G)$={B_i; B_i is a block of G}, and a, b be two variables taking values $+$ or $-$. The block-edge transformation graph G^{ab} is a graph whose vertex set is $E(G) \cup U(G)$, and two vertices x and y of G^{ab} are joined by an edge if and only if one of the following holds:

(i) Suppose x and y are in $E(G)$. $a = +$ if x, y are adjacent in G; $a = -$ if x and y are not adjacent in G.
(ii) Suppose $x \in E(G)$ and $y \in U(G)$. $b = +$ if x, y are incident with each other in G; $b = -$ if x, y are not incident with each other in G.

Thus, we obtain four kinds of block-edge transformation graphs G^{++}, G^{+-}, $G^{-+}$and G^{--} in which G^{++} is exactly the plick graph of G. Some other graph valued functions were studied in [1, 5, 8, 9, 11]. The vertex $e_i^{'}$ ($B_i^{'}$) of G^{ab} corresponding to edge e_i (block B_i) of G and is referred as edge (block)-vertex.

The following will be useful in the proof of our results.

Remark: 1.1 *$L(G)$ is an induced subgraph of G^{++} and G^{+-}.*

Remark: 1.2 *$J(G)$ is an induced subgraph of G^{-+} and G^{--}.*

Theorem: 1.1 [4] *If G is connected, then $L(G)$ is connected.*

Theorem: 1.2 [12] *Let G be a graph of size $q \geq 1$. Then $J(G)$ is connected if and only if G contains no edge that is adjacent to every other edge of G unless $G = K_4$ or C_4.*

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If a disconnected graph G has no isolated vertices, then clearly G contains no edge that is adjacent to every other edge of G. By Theorem 1.2, we have the following remark.

Remark: 1.3 *If a disconnected graph G has no isolated vertices, then $J(G)$ is connected.*

Since block-edge transformation graphs G^{ab} are defined on the edge set and block set of a graph G, isolated vertices of G (if G has) play no rule in G^{ab}. We assume that the graph G under consideration is nonempty and has no isolated vertices. In this paper, We investigate some basic properties of these four kinds of block-edge transformation graphs.

2. CONNECTEDNESS OF G^{ab}

The first theorem is well-known.

Theorem: 2.1 *For a given graph G, G^{++} is connected if and only if G is connected.*

Theorem: 2.2 *For a given graph G, G^{+-} is connected if and only if $G \neq B_i \cup B_j$ is not a block, where B_i and B_j are blocks.*

Proof: Suppose $G \neq B_i \cup B_j$ is not a block. Then we consider the following cases:

Case-1. Suppose G is connected. Then it has at least two blocks. Hence by Theorem 1.1 and Remark 1.1, $L(G)$ is a connected subgraph of G^{+-}, and also each block-vertex $B_i^{'}$ in G^{+-} is adjacent to at least one edge-vertex $e_j^{'}$, where e_j is not incident with B_i in G. Thus G^{+-} is connected.

Case-2. Suppose G is disconnected. Then it has at least three blocks. We see that in G^{+-}, each block-vertex $B_i^{'}$ is adjacent to at least two edge-vertices $e_j^{'}$, where e_j is not incident with B_i in G, and each edge-vertex $e_j^{'}$ is adjacent to edge-vertex $e_k^{'}$ and at least two block-vertices $B_i^{'}$ in G^{+-}, where e_k is adjacent to e_j, and B_i is not incident with e_j in G. Since in such a case, there is a path between any two vertices of G^{+-}. Hence G^{+-} is connected.

Conversely, suppose G^{+-} is connected. If G is a block, then $G^{+-} = L(G) \cup K_1$ is disconnected, a contradiction. If $G = B_i \cup B_j$, then G^{+-} is a disconnected graph having two components namely $L(B_i) + K_1$ and $L(B_j) + K_1$, a contradiction.

Theorem: 2.3 *For a given graph G, G^{-+} is connected if and only if G contains no block K_2 that is adjacent to every other edge of G.*

Proof: Suppose a graph G contains no block K_2 that is adjacent to every other edge of G. If G is a block, then $G^{-+} = J(G) + K_1$ is connected. If G has more than one block, then we consider the following two cases:

Case-1. If G contains no edge that is adjacent to every other edge of G, then by Remark 1.2 and Theorem 1.2, $J(G)$ is a connected subgraph of G^{-+}, and in G^{-+}, each block-vertex $B_i^{'}$ is adjacent to at least one edge-vertex $e_j^{'}$, where e_j is incident with B_i in G. Thus G^{-+} is connected.

Case-2. If G contains an edge e that is adjacent to every other edge of G, then clearly e is incident with a block B of size more than 2. And $(G-e)^{-+}$ is a connected subgraph of G^{-+} and $e^{'}, B^{'}, e_1^{'}$ is a path in G^{-+}(see fig. 1), where e_1 is incident with B, and each block-vertex $B_i^{'}$ in G^{-+} is adjacent to at least one edge-vertex $e_j^{'}$, where e_j is incident with B_i in G. Hence G^{-+} is connected.

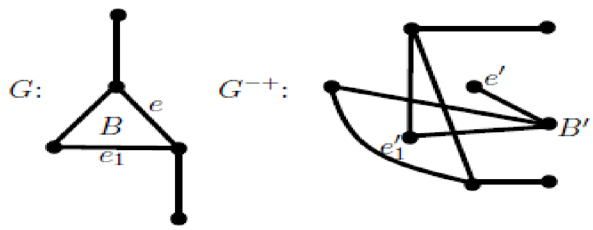

Figure 1

Conversely, suppose G^{-+} is connected. Assume G contains a block K_2, say e, that is adjacent to every other edge of G, then it is easy to see that G^{-+}=$(G-e)^{-+} \cup K_2$ is disconnected, a contradiction.

Theorem: 2.4 *For a given graph G, G^{--} is connected if and only if $G \neq P_3$ is not a block.*

Proof: Suppose $G \neq P_3$ is not a block. We consider the following two cases:

Case-1. Suppose G contains no edge that is adjacent to every other edge of G. Then by Remark 1.2 and Theorem 1.2, $J(G)$ is a connected subgraph of G^{--}, and each block-vertex B_i' is adjacent to at least one edge-vertex e_j' in G^{--}, where e_j is not incident with B_i in G. Thus G^{--} is connected.

Case-2. Suppose G contains an edge e that is adjacent to all other edge of G. Then by definition of G^{--}, each edge-vertex e_i' is adjacent to edge-vertex e_k' and at least one block-vertex B_j', where B_j is not incident with e_i, and e_k is not adjacent to e_i in G. And also each block-vertex B_j' is adjacent to at least one edge-vertex e_i', where e_i is not incident with B_j in G. Hence there is a path between any two vertices of G^{--}. Therefore G^{--} is connected.

Conversely, suppose G^{--} is connected. If G is a block, then $G^{--} = J(G) \cup K_1$ is disconnected, a contradiction. If $G = P_3$, then $G^{--} = 2K_2$ is disconnected, a contradiction.

3. GRAPH EQUATIONS AND ITERATIONS OF G^{ab}

For a given graph operator Φ, which graph is fixed under Φ ?, that is $\Phi(G) = G$. It is well known in [10] that for a given graph G, the interchange graph $G' = G$ if and only if G is a 2-regular graph.

For a given block-edge transformation graph G^{ab}, we define the iteration of G^{ab} as follows:

1. $G^{(ab)^1} = G^{ab}$
2. $G^{(ab)^n} = [G^{(ab)^{n-1}}]^{ab}$ for $n \geq 2$.

The isomorphism of G and G^{++} are shown in [6].

Theorem: 3.1 *The graphs G and G^{+-} are isomorphic if and only if $G = 2K_2$.*

Proof: Suppose $G^{+-} = G$. Assume $G \neq 2K_2$. We consider following two cases:

Case-1. Suppose G is a block. Then clearly $G^{+-} = L(G) \cup K_1$ is disconnected. Thus $G^{+-} \neq G$, a contradiction.

Case-2. Suppose G has at least two blocks with q edges. Then G^{+-} has at least $2q - 1$ edges. Hence the number of edges in G is less than that in G^{+-}. Thus $G^{+-} \neq G$, a contradiction.

Conversely, suppose $G = 2K_2$. Then it is easy to see that $G^{+-} = G$.

Corollary: 3.2 *The graphs G and $G^{(+-)^n}$ are isomorphic if and only if $G = 2K_2$.*

Theorem: 3.3 *The graphs G and G^{-+} are isomorphic if and only if $G = K_2$.*

Proof: Suppose $G^{-+} = G$. Assume $G \neq K_2$ with $p \geq 3$ vertices. We consider the following two cases:

Case-1. Suppose G is connected. We consider the following two subcases:

Subcase-1.1. Suppose G is a tree with p vertices. Then G has $p - 1$ edges and $p - 1$ blocks. Thus G^{-+} has $2p - 2$ vertices. Hence the number of vertices of G is less than that in G^{-+}. Therefore $G^{-+} \neq G$, a contradiction.

Subcase-1.2. Suppose G is not a tree with p vertices. Then G has at least p edges and at least one block. Thus G^{-+} has at least $p + 1$ vertices. Hence $G^{-+} \neq G$, a contradiction.

Case-2. Suppose G is a disconnected graph with q edges. Then G^{-+} has at least $q + 1$ edges. Hence $G^{-+} \neq G$, a contradiction.

Conversely, suppose $G = K_2$. Then clearly $G^{-+} = G$.

Corollary: 3.4 *The graphs G and $G^{(-+)^n}$ are isomorphic if and only if $G = K_2$.*

Theorem: 3.5 *For any graph* G, $G^{--} \neq G$.

Proof: If $G = K_2$, then $G^{--} = 2K_1 \neq G$. We consider the following two cases:

Case-1. Suppose $G \neq K_2$ is a connected graph. Since the definitions of G^{-+} and G^{--}, we have $|V(G^{-+})| = |V(G^{--})|$. By proof of the Theorem 3.3, we have $|V(G)| \neq |V(G^{-+})|$. Hence $|V(G)| \neq |V(G^{--})|$. Therefore $G^{--} \neq G$.

Case-2. Suppose G is a disconnected graph with q edges. Then G^{--} has at least $q+1$ edges. Hence $|E(G)| \neq |E(G^{--})|$. Therefore $G^{--} \neq G$. From all the above two cases, we have $G^{--} = G$.

Corollary: 3.6 *For any graph* G, $G^{(--)^n} \neq G$.

4. DIAMETERS OF G^{ab}

The distance between two vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of the shortest path between the vertices v_i and v_j in G. The shortest $v_i - v_j$ path is often called *geodesic*. The *diameter* of a connected graph G, denoted by $diam(G)$, is the length of any longest geodesic.

In this section, we consider the diameters of G^{ab}.

Theorem: 4.1 *If* G *is a connected graph, then* $diam(G^{++}) \leq diam(G) + 1$.

Proof: Let G be a connected graph. We consider the following three cases:

Case-1. Assume G is a tree. Then it is easy to see that $diam(G^{++}) = diam(G) + 1$.

Case-2. Assume G is a cycle C_n for $n \geq 3$. Then $G^{++} = W_{n+1}$ and $diam(G^{++}) < diam(G) + 1$.

Case-3. Assume G contains a cycle C_n for $n \geq 3$. Corresponding to cycle C_n, W_{n+1} appears as subgraph in G^{++}. Therefore $diam(G^{++}) \leq diam(G) + 1$.

From all the above three cases, we have $diam(G^{++}) \leq diam(G) + 1$.

Theorem: 4.2 *If a graph* G *has at least three blocks, then*

$$diam(G^{+-}) = \begin{cases} 2 \; if \; every \; component \; of \; G \; has \; at \; least \; one \; cutvertex \\ 3 \; if \; at \; least \; one \; component \; of \; G \; is \; a \; block. \end{cases}$$

Proof: Let e_1', e_2' be the two edge-vertices of G^{+-}. If e_1 and e_2 are adjacent edges in G, then e_1' and e_2' are adjacent in G^{+-}. If e_1 and e_2 are not adjacent edges in G, then there exists a block B which is incident with neither e_1 nor e_2 in G such that e_1', B', e_2' is a path of length 2 in G^{+-}.

Let B_1', B_2' be the two block-vertices of G^{+-}. Then there exists an edge e which is incident with neither B_1 nor B_2 in G such that B_1', e', B_2' is a path in G^{+-} of length 2.

Let e' and B' be the edge-vertex and block-vertex of G^{+-} respectively. If e is not incident with B in G, then e' and B' are adjacent in G^{+-}. If e is incident with B in G, then we consider the following two cases:

Case-1. If every component of G has at least one cutvertex, then there exists an edge e_1 which is adjacent to e, and is not incident with B such that e', e_1', B' is a path of length 2 in G^{+-}.

Case-2. If at least one component of G is a block, say B, then there exists not incident block B_1 and edge e_1, where B_1 is not incident with e, and e_1 is incident with neither B nor B_1 such that e', B_1', e_1', B' is a path in G^{+-} of length 3.

Theorem: 4.3 *If a connected graph* G *has two blocks, then* $diam(G^{+-}) \leq 5$.

Proof: Suppose G is a connected graph with two blocks B_1 and B_2 of size q_1 and q_2 respectively. Then K_{1,q_1} and K_{1,q_2} are two edge-disjoint subgraphs of G^{+-}. And there exists at least one edge e' in G^{+-} is incident with exactly one pendant vertex of K_{1,q_1} and K_{1,q_2}. It is easy that see that the diameter of star is at most 2.

Hence $diam(G^{+-}) = diam(K_{1,q_1}) + diam(K_{1,q_2}) + 1 \leq 2 + 2 + 1 = 5$.

Theorem: 4.4 *If a graph G contains no block K_2 that is adjacent to other edge of G, then $daim(G^{-+}) \leq 5$.*

Proof: For $e_1^{'}$, $e_2^{'}$ be the two edge-vertices of G^{-+}. If e_1 and e_2 are not adjacent edges in G, then $e_1^{'}$ and $e_2^{'}$ are adjacent in G^{-+}. If e_1 and e_2 are adjacent edges in G, then we have one of the following case:

Case-1. If e_1 and e_2 are incident with same block B, then $e_1^{'}, B^{'}, e_2^{'}$ is a path of length 2 in G^{-+}.

Case-2. If e_1 and e_2 are incident with different blocks B_1 and B_2 respectively, then we have the following subcases:

Subcase-2.1. If there is an edge e which is adjacent to neither e_1 nor e_2 in G, then $e_1^{'}, e^{'}, e_2^{'}$ is a path in G^{-+} of length 2.

Subcase-2.2. If there is an edge e which is incident with B_2, and is not adjacent to e_1, then $e_1^{'}, e^{'}, B_2^{'}, e_2^{'}$ is a path in G^{-+} of length 3.

Subcase-2.3. If there are two not adjacent edges e_3 and e_4, where e_3 and e_4 are not adjacent to e_1 and e_2 respectively, then $e_1^{'}, e_3^{'}, e_4^{'}, e_2^{'}$ is a path in G^{-+} of length 3.

For $B_1^{'}$, $B_2^{'}$ be the two block-vertices of G^{-+}. Let e_1 and e_2 be the two edges incident with the blocks B_1 and B_2 respectively. We have the following cases:

Case-1. If e_1 and e_2 are not adjacent edges in G, then $B_1^{'}, e_1^{'}, e_2^{'}, B_2^{'}$ is a path of length 3 in G^{-+}.

Case-2. If e_1 and e_2 are adjacent edges in G, then we have the following subcases:

Subcase-2.1. If there is an edge e which is adjacent to neither e_1 nor e_2 in G, then $B_1^{'}, e_1^{'}, e^{'}, e_2^{'}, B_2^{'}$ is a path of length 4 in G^{-+}.

Subcase-2.2. If there are two not adjacent edges e_3 and e_4, where e_3 and e_4 are not adjacent to e_2 and e_1 respectively, then $B_1^{'}, e_1^{'}, e_4^{'}, e_3^{'}, e_2^{'}, B_2^{'}$ is a path in G^{-+} of length 5.

For $e_1^{'}$ and $B_2^{'}$ be the edge-vertex and block-vertex of G^{-+} respectively. If e_1 is incident with B_2 in G, then $e_1^{'}$ and $B_2^{'}$ are adjacent in G^{-+}. If e_1 is not incident with B_2 in G, then we have the following cases:

Case-1. If there is an edge e_2 is incident with B_2, where e_2 is not adjacent to e_1 in G, then $B_2^{'}, e_2^{'}, e_1^{'}$ is a path in G^{-+} of length 2.

Case-2. If there is an edge e_2 is incident with B_2, and is adjacent to an edge e in G, where e_1 and e are incident with B_1, then $B_2^{'}, e_2^{'}, e^{'}, B_1^{'}, e_1^{'}$ is a path of length 4 in G^{-+}.

Case-3. If there is an edge e which is adjacent to neither e_1 nor e_2 , and e_2 is incident with B_2, then $B_2^{'}, e_2^{'}, e^{'}, e_1^{'}$ is a path of length 3 in G^{-+}.

Theorem: 4.5 *If a graph $G \neq P_3$ is not a block, then $diam(G^{--}) \leq 4$.*

Proof: Let $e_1^{'}$, $e_2^{'}$ be the two edge-vertices of G^{--}. If e_1 and e_2 are not adjacent edges in G, then $e_1^{'}$ and $e_2^{'}$ are adjacent in G^{--}. If e_1 and e_2 are adjacent edges in G, then we have one of the following case:

Case-1. If e_1 and e_2 are incident with same block, then there exist a block B which is incident with neither e_1 nor e_2 such that $e_1^{'}, B^{'}, e_2^{'}$ is a path of length 2 in G^{--}.

Case-2. If e_1 and e_2 are incident with different blocks B_1 and B_2 respectively in G, then we have the following subcases:

Subcase-2.1. If there is a block B which is incident to neither e_1 nor e_2 in G, then $e_1^{'}, B^{'}, e_2^{'}$ is a path in G^{--} of length 2.

Subcase-2.2. If there is an edge e is incident with block B_2, and is not adjacent to e_1, then $e_2^{'}, B_1^{'}, e^{'}, e_1^{'}$ is a path in G^{--} of length 3.

Subcase-2.3. If there is an edge e_3 which is adjacent to neither e_1 nor e_2, then $e_1^{'}, e_3^{'}, e_2^{'}$ is a path in G^{--} of length 2.

Let B_1', B_2' be two block-vertices of G^{--}. We have the following cases:

Case-1. If there is an edge e which is incident with neither B_1 nor B_2, then B_1', e', B_2' is a path of length 2 in G^{--}.

Case-2. If there are two not adjacent edges e_1 and e_2 are incident with B_1 and B_2 respectively, then B_1', e_2', e_1', B_2' is a path of length 3 in G^{--}.

Let e' and B' be the edge-vertex and block-vertex of G^{--} respectively. If e is not incident with B in G, then e' and B' are adjacent in G^{--}. If e is incident with B in G, then we have the following cases:

Case-1. If there is an edge e_1 is incident with B, and is not adjacent to edge e in G, then e', e_1', B' is a path in G^{--} of length 2.

Case-2. If there are two not adjacent edges e_1 and e_2, where e_1 is not incident with B, and e_2 is not adjacent to e, then B', e_1', e_2', e' is a path of length 3 in G^{--}.

Case-3. If there are not incident edge e_2 and block B_3, where e_2 is not incident with B, and B_3 is not incident to e, then e', B_3', e_2', B' is a path of length 3 in G^{--}.

Case-4. If there is an edge e_1 which is incident with B_1, and is not adjacent to an edge e_2, where e_2 is incident with B, then B', e_1', e_2', B_1', e' is a path of length 4 in G^{--}.

5. ACKNOWLEDGEMENT

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