International Research Journal of Pure Algebra-5(6), 2015, 94-100
Available online through www.rjpa.info ISSN 2248–9037

ON ESSENTIAL PSEUDO *P*- INJECTIVE MODULE

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(Received On: 15-06-15; Revised & Accepted On: 25-06-15)

ABSTRACT

A module M is said to be an essential pseudo p-injective module if every monomorphism $f: N \to M$ extends to M, where $N \in \mathcal{F}$, the set of all essential cyclic submodules of M. We establish several equivalent conditions for a module to be pseudo p −injective. We show that if a module has no proper essential submodule, then it is an essential Pseudo p −injective module. We prove that an essential pseudo p −injective module having no proper essential submodule is isomorphic to its direct summand. We also show that an essential submodule of an essential pseudo p −injective module is also essential pseudo p −injective and essential pseudo stable under certain conditions. Moreover, every essential pseudo stable submodule of an essential pseudo p −injective module, M is also essential pseudo p −injective and intersection of any two invariant submodules of M is an essential pseudo stable submodule of M.

Keywords: *Essential Pseudo p −injective module, Pseudo Stable Submodule, Direct Summand.*

1. INTRODUCTION

Pseudo injective modules have been studied by several authors. Dinh and Loperge [3] have studied pseudo injective modules and discussed some connections between pseudo injective rings and notions of equivalence of codes over finite rings. Dinh [2] also discussed sufficient conditions for a pseudo p–injective module to be quasi-injective. Jain, Singh [8] and Teply [10] have discussed several examples of pseudo-injective module to be quasi-injective. Tiwary [11] has generalized projective modules to small projective modules. Recently, Talebi [9] discussed pseudo projectivity and small pseudo projectivity relative to a module. Bharadwaj [1] has also generalized pseudo projective modules to small pseudo projective modules.

In this paper, our attempt is to study the dualized notion of small projective modules. We define essential pseudo p −injective modules and study various characteristics of these modules. A module M is said to be an essential pseudo p-injective module if every monomorphism $f: N \to M$ extends to M, where $N \in \mathcal{F}$, the set of all essential cyclic submodules of M. We establish several equivalent conditions for a module to be pseudo p −injective. We show that if a module has no proper essential submodule, then it is an essential Pseudo p −injective module.

We prove that an essential pseudo p −injective module having no proper essential submodule is isomorphic to its direct summand. We also show that an essential submodule of an essential pseudo p −injective module is also essential pseudo p −injective and essential pseudo stable under certain conditions. Moreover, every essential pseudo stable submodule of an essential pseudo p −injective module, M is also essential pseudo p −injective and intersection of any two invariant submodules of M is an essential pseudo stable submodule of M.

2. DEFINITIONS AND NOTATIONS

In this section we define the basic terms that are needed for the sequel. Throughout our discussion R denotes a ring with unity and M denotes a right R −module.

Definition 2.1: A sub module C of M is said to be *essential* if it has non–zero intersection with every non- zero sub module of M. Thus, if B is any non zero sub module of M then $B \cap C \neq 0$.

Definition 2.2: An R −module M is said to be *uniform*, if all submodules of M are essential in M.

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Definition 2.3: An R −module M is said to be *essential pseudo* p *−injective* if every monomorphism $f: N \rightarrow M$ extends to M, where $N \in \mathcal{F}$, set of all essential cyclic submodules of M.

Definition 2.4: A submodule N of M is called stable if $f(N) \subseteq N$, for each R −homomorphism $f: N \rightarrow M$.

Definition 2.5: A submodule N of M is said to be *essential pseudo stable* if for any monomorphisms $f, g: mR \rightarrow M$ there exists $h \in End\ (M)$ such that $f = h \circ g$, then $h(N) \subseteq N$.

3. MAIN RESULTS

In this section we study the main results on essential pseudo p −injective module.

Theorem 3.1: Let M be essential pseudo P −injevtive module with $r_R(m) = r_R(n)$, $m, n(\neq 0) \in M$, then $Ann_M\big(Ann_R(m)\big) = m\,S$, $m \in M$ such that $mR \in \mathcal{F}$ where $S = End_R(M)$.

***Proof*:** Since $Ann_R(m)m = 0$, $\forall\, m \in M$

$\Rightarrow s(Ann_R(m)m) = 0, \qquad s \in S$

$\Rightarrow Ann_R(m)s(m) = 0$

$\Rightarrow s(m) \in Ann_M(Ann_R(m))$

$\Rightarrow m\,s \in Ann_M\big(Ann_R(m)\big)$

$\Rightarrow m\,S \subseteq Ann_M\ (Ann_R(m))$

Again, suppose $m' \in Ann_M(Ann_R(m))$.

Since $r_R(m) = r_R(n)$, $\forall\, m, n \in M$ then the mapping $g : mR \rightarrow M$ such that $g(mr) = m'r$. Then g is a monomerphism. By essential pseudo P −injevtivity of M, g extends to an endomorphism $s \in End_R(M)$. Therefore,

$$m' = g(m) = s(m) = ms \subseteq mS$$

Thus $Ann_M(Ann_R(m) = m\,S$, $\forall m \in M$

Theorem 3.2: Let M be an essential pseudo P −injevtive module. Then $Ann_R(a) = Ann_R(b) \Rightarrow bS = aS$, where $a, b \in M$ such that $a, b \in \mathcal{F}$ and $S = End_R(M)$.

***Proof*:** Given M is essential pseudo P −injevtive module and

$$Ann_R\ (a) = Ann_R\ (b)$$

Let $x \in bS \Rightarrow x \in Ann_M\big(Ann_R(b)\big)$ by theorem 3.1.

$\Rightarrow Ann_R(b)\, x = 0$

$\Rightarrow Ann_R\ (a)x = 0$

$\Rightarrow x \in Ann_M\ (Ann_R\ (a))$

$\Rightarrow x \in aS$ by theorem 3.1.

Thus $bS \subseteq aS$.

Similarly, $aS \subseteq bS$.

Thus $a\,S = bS$.

Theorem 3.3: An R −module M is essential pseudo P −injective module if and only if for any monomorphisms $\alpha, \beta: mR \rightarrow M$ where $mR \in \mathcal{F}$ then there exists $\gamma \in S = End_R(M)$ such that $\gamma \circ \beta = \alpha$.

Proof: Let M be a pseudo P −injevtive module. Let $\alpha, \beta: mR \rightarrow M$ be monomorphisms.

First we show $Ann_R(\beta\, m) = Ann_R(\alpha\, m)$

Let $x \in Ann_R(\beta\, m) \Rightarrow (x)\beta\, m = 0$

$\Rightarrow \beta\ (x\, m) = 0 \Rightarrow x\, m = 0$ (since β is one-one)

$\Rightarrow \alpha(xm) = 0 \Rightarrow x\alpha(m) = 0$

$\Rightarrow x \in Ann_R\ (\alpha\, m)$

Thus $Ann_R\ (\beta\, m) \subseteq Ann_R\ (\alpha\, m)$

Similarly, $Ann_R\ (\alpha\ m) \subseteq\ Ann_R\ (\beta\ m)$

Thus $Ann_R\ (\beta\ m) = Ann_R\ (\alpha\ m) \Rightarrow (\beta\ m)S\ =\ (\alpha\ m)S$ (by theorem 3.2.)
$\Rightarrow\ \beta S\ =\ \alpha S$

So $\exists\ \ \gamma \in S = End_R(M)$ such that $\gamma \circ \beta\ =\ \alpha$

Conversely, let $\alpha\colon mR \rightarrow\ M$ be a monomorphism.

We consider $\beta\colon mR \rightarrow\ M$ such that $\beta(mr) = mr$

Then β is a monomorphism.

By the given condition $\exists\ \ \gamma \in S$ such that $\gamma \circ \beta = \alpha$

Let $x \in mR \Rightarrow x = mr,$ for some $r \in R$

Thus $\gamma\ (x) = \gamma\ (mr) = \gamma\ \beta(mr) = \alpha(mr) = \alpha(x)$

Hence γ is an extension of α.

Therefore M is essential pseudo P −injective module.

Theorem 3.4: Let M be an essential pseudo P −injective module. If $\alpha \in S$ and $m \in M$ such that $mR \in \mathcal{F}$, then $Ann_S[Ker\alpha \cap mR]\ = \alpha\ S + Ann_S(m)$.

Proof: First we show

$$\alpha\ S + Ann_S(m) \subseteq\ Ann_S[Ker\alpha \cap mR]$$

Let $x \in Ker\alpha \cap mR$
$\Rightarrow\ x \in Ker\alpha$ and $x \in mR$
$\Rightarrow\ \alpha x = 0$ and $x = mr,\quad r \in R$

Let $f \in Ann_S(m) \Rightarrow mf = 0\ \Rightarrow r(mf) = 0$
$\Rightarrow (\ mr)f = 0\ \Rightarrow xf = 0$

Then $x[\alpha\ S\ +\ Ann_S(m)] = x\ \alpha\ S\ +\ x\ Ann_S(m)$
$= \alpha\ x\ S +\ x\ Ann_S(m)$
$= 0\ +\ 0\ =\ 0$

Therefore, $\alpha\ S\ +\ Ann_S(m) \subseteq\ Ann_S[Ker\alpha\ \cap\ mR]$

Next, $Ann_S[Ker\alpha\ \cap mR] \subseteq \alpha\ S\ +\ Ann_S(m)$

Let $\beta \in Ann_S[Ker\alpha\ \cap\ mR]$

We claim that $Ann_R(\alpha\ m)\ =\ Ann_R(\beta\ m)$

Let $x \in Ann_R(\alpha\ m) \Rightarrow x(\alpha\ m) = 0$
$\Rightarrow \alpha\ (x\ m) = 0\ \ \Rightarrow\ \alpha\ (m\ x) = 0$
$\Rightarrow m\ x \in Ker\alpha \cap mR$

Therefore $(mx)\beta\ = 0\ \ \Rightarrow\ (x\ m)\beta = 0$
$\Rightarrow x\ \beta(m) = 0$
$\Rightarrow x \in\ Ann_R(\beta m)$

Therefore, $Ann_R(\alpha\ m) \subseteq\ Ann_R(\beta\ m)$

Similarly $Ann_R(\beta\ m) \subseteq\ Ann_R(\alpha\ m)$

Therefore, $Ann_R(\alpha\ m) =\ Ann_R(\beta\ m)$
$\Rightarrow \alpha m\ S = \beta m\ S$

Then $\exists\ \gamma \in S$ such that $(\alpha \circ \gamma)m = \beta\, m$

$\Rightarrow (\beta - \alpha \circ \gamma)m = 0$

$\Rightarrow \beta - \alpha \circ \gamma \in Ann_S(m)$

So $\beta \in \alpha S + Ann_S(m)$

Thus $Ann_S[Ker\ \alpha \cap mR] = \alpha S + Ann_S(m)$.

Theorem 3.5: For a uniform module M the following conditions are equivalent:
(i) M is essential pseudo $p -$ injective.
(ii) M is pseudo $p -$ injective.

Proof: (i) $\Rightarrow$ (ii):

Let M be an essential pseudo p-injective module.

We are to show that M is pseudo p- injective.

Let $f, g: mR \to M$ be monomorphisms where $mR \in \mathcal{F}$.

Given M is a uniform module, therefore all submodules of M are essential in M. So by the definition of essential pseudo $p -$ injectivity of M, there exists $h \in End(M)$ such that $f = h \circ g$.

Thus, M is pseudo $p -$ injective.

Next (ii) $\Rightarrow$ (i): Follows from definition.

Note: If M has no essential extension then M is essential pseudo $p -$injective.

Theorem 3.6: Let M be an essential pseudo $p -$injective module then M has no proper essential submodule $K \in \mathcal{F}$ where K is isomorphic to a direct summand of M.

***Proof*:** Let Q be a direct summand of M and K be any essential submodule of M such that $K \cong Q$.

Let $\varphi: K \to Q$ be the isomorphism, $\gamma: K \to M$ be the natural map, $j_Q: Q \to M$ be the injection map and $\pi_Q: M \to Q$ be the projection mapping.

Since M is essential pseudo $p -$ injective then there exists $h \in End(M)$ such that the following diagram commutes.

$$\begin{tikzcd}
M & \\
Q \arrow[u, "j_Q"] & \\
K \arrow[u, "\varphi"] \arrow[r, "\gamma"'] & M \arrow[ul, dashed, "\beta"'] \arrow[uul, dashed, "h"']
\end{tikzcd}$$

Thus $h \circ \gamma = j_Q \circ \varphi$.

Again define $\beta : M \to Q$ by $\beta = \pi_Q \circ h$

And $\gamma' : M \to K$ by $\gamma(m) = \varphi^{-1} \circ \beta(m)$

Now $\gamma' \circ \gamma \; = \; \varphi^{-1} \circ \pi_Q \circ h \circ \gamma$
$= \varphi^{-1} \circ \pi_Q \circ J_Q \circ \varphi$
$= \varphi^{-1} \circ \varphi$
$= I_K$

Thus the sequence $0 \to K \to M \to Q \to 0$ splits and therefore K is a direct summand of M. So $\exists\; N \subseteq M$ such that $M = K \oplus N$.

Now $M = K + N$ and $K \cap N = 0 \Rightarrow K = 0$

Hence M has no proper essential submodule satisfying the given condition.

Theorem 3.7: Let M be an essential pseudo p-injective module and $\varphi: mR \to M$ be any monomorphism then there exists a mono-endomorphism $h \in End\,(M)$ such that $Im\,\varphi = Im\,(h \circ \varphi)$ is stable under h.

***Proof*:** Given, M is an essential pseudo $p-$injective module. The monomorphism $\varphi: mR \to M$ induces an isomorphism $\varphi *: mR \to Im\varphi$.

Let $f: Im\,\varphi \to M$ be the natural map. Then by essential pseudo $p-$injectivity of M, there exists $h \in End\,(M)$ such that the following diagram commutes.

$$\begin{tikzcd} & M \arrow[dr, dashed, "h"] & \\ N \arrow[ur, "\varphi"] \arrow[r, "\varphi *"'] & Im\,\varphi \arrow[r, "f"'] & M \end{tikzcd}$$

Thus $f \circ \varphi * = h \circ \varphi$.

Now, we are to show h is one-one.

Let $x \in Im\,\varphi \cap Ker\,h$

Then $x \in Im\,\varphi$ and $x \in Ker\,h$
$\Rightarrow x = \varphi\,(mr)$, for some $r \in R$ and $h(x) = 0$

$\Rightarrow h(x) = h(\varphi(mr))$
$= (h \circ \varphi)(mr)$
$= (f \circ \varphi *)\,(mr)$
$= f\,(\varphi * (mr))$
$= f\,(\varphi\,(mr)$, as φ induce φ ∗.

$\Rightarrow f(\varphi(mr)) = 0$
$\Rightarrow \varphi\,(mr) = 0$
$\Rightarrow x = 0.$

Therefore, $Im\,\varphi \cap Ker\,h = 0$
$\Rightarrow Ker\,h = 0$ as Im φ is essential in M.
$\Rightarrow h$ is one-one.

Now, let $x \in Im\,\varphi \Rightarrow x \in Im\,f \circ \varphi *$
$\Rightarrow f \circ \varphi * (x) = m$, for some $m \in M$.
$\Rightarrow (h \circ \varphi)(x) = m$
$\Rightarrow x \in Im\,(h \circ \varphi)$
$\Rightarrow Im\,\varphi \subseteq Im(h \circ \varphi)$

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Again, let $y \in Im\ (h \circ \varphi)$
$\Rightarrow (h \circ \varphi)(y) = m$
$\Rightarrow (f \circ \varphi *)(y) = m$
$\Rightarrow y \in Im\ (f \circ \varphi *)$
$\Rightarrow y \in Im\ \varphi$

Therefore $Im\ (h \circ \varphi) \subseteq Im\ \varphi$
$\Rightarrow h\ (Im\ \varphi\ ) \subseteq Im\varphi$.

Thus, $Im\ \varphi$ is invariant under mono – endomorphism of M.

Theorem 3.8: Let M be an essential pseudo p −injective module and $K \in \mathcal{F}$ be a submodule of M, then K is essential pseudo p −injective if K is stable under mono endomorphism of M.

Proof: Let $\gamma: K \to M$ be the natural map $f, g: mR \to K$ be a monomorphisms.

Then by essential pseudo p − injectivity of M, there exists $\varphi \in End(M)$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & K \arrow[r, "\gamma"] & M \\
 & K \arrow[u, "\psi"] \arrow[r, "\gamma"'] & M \arrow[u, "\varphi"']
\end{tikzcd}$$

Thus $\gamma \circ f = \varphi \circ \gamma \circ g$.

We define $\psi: K \to K$ as $\psi(x) = \varphi\ (x),\ x \in K$.

Then, ψ is well defined as let $x_1 = x_2 \Rightarrow \varphi(x_1) = \varphi(x_2) \Rightarrow \psi(x_1) = \psi(x_2)$.

Thus ψ is well defined and

$$\gamma \circ \psi = \varphi \circ \gamma \Rightarrow \gamma \circ \psi \circ g = \varphi \circ \gamma \circ g \Rightarrow \gamma \circ \psi \circ g = \gamma \circ f$$

$\Rightarrow \psi \circ g = f$, since γ is natural map.

Hence, K is essential pseudo p − injective.

Theorem 3.9: Let M be an essential pseudo p −injective module and $g: N \to M$ be any monomorphism, where $N \in \mathcal{F}$. Then N is essential pseudo p − injective.

Proof: Given, M is aessential pseudo p −injective module and $g: N \to M$ be any monomorphism. Then by theorem 3.7, $Im\ g = Im(h \circ g)$ is stable under h.

But $Im(g)$ is a submodule of M.

Then by theorem 3.8, we have $Im\ g$ is essential pseudo p −injective module.

Since $Im\ g \cong N$ and $Im\ g$ is essential pseudo p − injective, therefore N is essential pseudo p − injective module.

Theorem 3.10: Let M be an essential pseudo p −injective module and $N \in \mathcal{F}$ be submodule of M stable under mono-endomorphism of M, then N is essential pseudo stable.

Proof: Given, M is an essential pseudo p −injective module and N is an essential cyclic submodule of M stable under mono–endomorphism of M, then by theorem 3.8, N is essential pseudo p −injective.

Also as N is stable under mono-endomorphism of M, we have $h\ (N) \subseteq N$, where $h \in End(M)$. Thus N is essential pseudo stable.

Theorem 3.11: Let M be an essential pseudo p −injective module and T be an essential pseudo stable sub module of M, then T is essential pseudo p −injective.

Proof: Let $g, f:\ mR \ \rightarrow\ T$ be any monomorphisms and $\gamma: T \rightarrow\ M$ be the natural map.

Then by essential pseudo $p-$ injectivity of M, there exists $h \in End\ (M)$ such that $\gamma \circ f = h \circ \gamma \circ g$.

Since T is essential pseudo stable, therefore $h(\ T\ ) \subseteq T$.

Hence by theorem 3.8, T is essential pseudo $p-$ injective.

Theorem 3.12: If M is an essential pseudo p −injective module and $\varphi: mR \rightarrow M$ is any monomorphism, then $Im\ \varphi = K$ is an essential pseudo stable submodule of M.

Proof: Here $\varphi: mR\ \rightarrow\ M$ induces an isomorphism $\varphi *: mR\ \rightarrow Im\varphi$.

$$\begin{tikzcd}
 & M \arrow[drr, dashed, "h"] & & \\
N \arrow[ur, "\varphi"] \arrow[r, "\varphi *"'] & Im\,\varphi \arrow[rr, "f"'] & & M
\end{tikzcd}$$

Let $f: Im\varphi\ \rightarrow\ M$ be the natural map. Since, M is essential pseudo $p-$ injective module therefore there exists $h\ \in End\ (M)$ such that the following diagram commutes.

Thus $f \circ \varphi * = h \circ \varphi$.

Now, let $h(K) \not\subset K$. Then there exists $k \in K$ such that $h(k) \in h(K)$ and $h(k) \notin K$.

Then $0 \neq h(k) = h\left(\varphi\ (k')\right) = (h \circ \varphi)(k') = (\ f \circ \varphi *)(k') \in Im(\ f \circ \varphi *)$ as $K \subseteq Im\ f \circ \varphi *$,
which is a contradiction.

Thus, h (K) $\subseteq$ K

$\Rightarrow$ K is essential pseudo stable.

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Source of Support: Nil, Conflict of interest: None Declared

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