

ON ESSENTIAL PSEUDO P - INJECTIVE MODULE

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ABSTRACT

A module M is said to be an essential pseudo p -injective module if every monomorphism $f: N \rightarrow M$ extends to M , where $N \in \mathcal{F}$, the set of all essential cyclic submodules of M . We establish several equivalent conditions for a module to be pseudo p -injective. We show that if a module has no proper essential submodule, then it is an essential Pseudo p -injective module. We prove that an essential pseudo p -injective module having no proper essential submodule is isomorphic to its direct summand. We also show that an essential submodule of an essential pseudo p -injective module is also essential pseudo p -injective and essential pseudo stable under certain conditions. Moreover, every essential pseudo stable submodule of an essential pseudo p -injective module, M is also essential pseudo p -injective and intersection of any two invariant submodules of M is an essential pseudo stable submodule of M .

Keywords: Essential Pseudo p -injective module, Pseudo Stable Submodule, Direct Summand.

1. INTRODUCTION

Pseudo injective modules have been studied by several authors. Dinh and Loperge [3] have studied pseudo injective modules and discussed some connections between pseudo injective rings and notions of equivalence of codes over finite rings. Dinh [2] also discussed sufficient conditions for a pseudo p -injective module to be quasi-injective. Jain, Singh [8] and Teply [10] have discussed several examples of pseudo-injective module to be quasi-injective. Tiwary [11] has generalized projective modules to small projective modules. Recently, Talebi [9] discussed pseudo projectivity and small pseudo projectivity relative to a module. Bharadwaj [1] has also generalized pseudo projective modules to small pseudo projective modules.

In this paper, our attempt is to study the dualized notion of small projective modules. We define essential pseudo p -injective modules and study various characteristics of these modules. A module M is said to be an essential pseudo p -injective module if every monomorphism $f: N \rightarrow M$ extends to M , where $N \in \mathcal{F}$, the set of all essential cyclic submodules of M . We establish several equivalent conditions for a module to be pseudo p -injective. We show that if a module has no proper essential submodule, then it is an essential Pseudo p -injective module.

We prove that an essential pseudo p -injective module having no proper essential submodule is isomorphic to its direct summand. We also show that an essential submodule of an essential pseudo p -injective module is also essential pseudo p -injective and essential pseudo stable under certain conditions. Moreover, every essential pseudo stable submodule of an essential pseudo p -injective module, M is also essential pseudo p -injective and intersection of any two invariant submodules of M is an essential pseudo stable submodule of M .

2. DEFINITIONS AND NOTATIONS

In this section we define the basic terms that are needed for the sequel. Throughout our discussion R denotes a ring with unity and M denotes a right R -module.

Definition 2.1: A sub module C of M is said to be *essential* if it has non-zero intersection with every non-zero sub module of M . Thus, if B is any non zero sub module of M then $B \cap C \neq 0$.

Definition 2.2: An R -module M is said to be *uniform*, if all submodules of M are essential in M .

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Definition 2.3: An R –module M is said to be *essential pseudo p –injective* if every monomorphism $f: N \rightarrow M$ extends to M , where $N \in \mathcal{F}$, set of all essential cyclic submodules of M .

Definition 2.4: A submodule N of M is called *stable* if $f(N) \subseteq N$, for each R –homomorphism $f: N \rightarrow M$.

Definition 2.5: A submodule N of M is said to be *essential pseudo stable* if for any monomorphisms $f, g: mR \rightarrow M$ there exists $h \in \text{End}(M)$ such that $f = h \circ g$, then $h(N) \subseteq N$.

3. MAIN RESULTS

In this section we study the main results on essential pseudo p –injective module.

Theorem 3.1: Let M be essential pseudo P –injective module with $r_R(m) = r_R(n)$, $m, n (\neq 0) \in M$, then $\text{Ann}_M(\text{Ann}_R(m)) = mS$, $m \in M$ such that $mR \in \mathcal{F}$ where $S = \text{End}_R(M)$.

Proof: Since $\text{Ann}_R(m)m = 0$, $\forall m \in M$
 $\Rightarrow s(\text{Ann}_R(m)m) = 0$, $s \in S$
 $\Rightarrow \text{Ann}_R(m)s(m) = 0$
 $\Rightarrow s(m) \in \text{Ann}_M(\text{Ann}_R(m))$
 $\Rightarrow m s \in \text{Ann}_M(\text{Ann}_R(m))$
 $\Rightarrow mS \subseteq \text{Ann}_M(\text{Ann}_R(m))$

Again, suppose $m' \in \text{Ann}_M(\text{Ann}_R(m))$.

Since $r_R(m) = r_R(n)$, $\forall m, n \in M$ then the mapping $g: mR \rightarrow M$ such that $g(mr) = m'r$. Then g is a monomorphism. By essential pseudo P –injectivity of M , g extends to an endomorphism $s \in \text{End}_R(M)$. Therefore,
 $m' = g(m) = s(m) = ms \subseteq mS$

Thus $\text{Ann}_M(\text{Ann}_R(m)) = mS$, $\forall m \in M$

Theorem 3.2: Let M be an essential pseudo P –injective module. Then $\text{Ann}_R(a) = \text{Ann}_R(b) \Rightarrow bS = aS$, where $a, b \in M$ such that $a, b \in \mathcal{F}$ and $S = \text{End}_R(M)$.

Proof: Given M is essential pseudo P –injective module and
 $\text{Ann}_R(a) = \text{Ann}_R(b)$

Let $x \in bS \Rightarrow x \in \text{Ann}_M(\text{Ann}_R(b))$ by theorem 3.1.
 $\Rightarrow \text{Ann}_R(b)x = 0$
 $\Rightarrow \text{Ann}_R(a)x = 0$
 $\Rightarrow x \in \text{Ann}_M(\text{Ann}_R(a))$
 $\Rightarrow x \in aS$ by theorem 3.1.

Thus $bS \subseteq aS$.

Similarly, $aS \subseteq bS$.

Thus $aS = bS$.

Theorem 3.3: An R –module M is essential pseudo P –injective module if and only if for any monomorphisms $\alpha, \beta: mR \rightarrow M$ where $mR \in \mathcal{F}$ then there exists $\gamma \in S = \text{End}_R(M)$ such that $\gamma \circ \beta = \alpha$.

Proof: Let M be a pseudo P –injective module. Let $\alpha, \beta: mR \rightarrow M$ be monomorphisms.

First we show $\text{Ann}_R(\beta m) = \text{Ann}_R(\alpha m)$

Let $x \in \text{Ann}_R(\beta m) \Rightarrow (x)\beta m = 0$
 $\Rightarrow \beta(xm) = 0 \Rightarrow xm = 0$ (since β is one-one)
 $\Rightarrow \alpha(xm) = 0 \Rightarrow x\alpha(m) = 0$
 $\Rightarrow x \in \text{Ann}_R(\alpha m)$

Thus $\text{Ann}_R(\beta m) \subseteq \text{Ann}_R(\alpha m)$

Similarly, $Ann_R(\alpha m) \subseteq Ann_R(\beta m)$

Thus $Ann_R(\beta m) = Ann_R(\alpha m) \Rightarrow (\beta m)S = (\alpha m)S$ (by theorem 3.2.)
 $\Rightarrow \beta S = \alpha S$

So $\exists \gamma \in S = End_R(M)$ such that $\gamma \circ \beta = \alpha$

Conversely, let $\alpha: mR \rightarrow M$ be a monomorphism.

We consider $\beta: mR \rightarrow M$ such that $\beta(mr) = mr$

Then β is a monomorphism.

By the given condition $\exists \gamma \in S$ such that $\gamma \circ \beta = \alpha$

Let $x \in mR \Rightarrow x = mr$, for some $r \in R$

Thus $\gamma(x) = \gamma(mr) = \gamma\beta(mr) = \alpha(mr) = \alpha(x)$

Hence γ is an extension of α .

Therefore M is essential pseudo P -injective module.

Theorem 3.4: Let M be an essential pseudo P -injective module. If $\alpha \in S$ and $m \in M$ such that $mR \in \mathcal{F}$, then $Ann_S[Ker\alpha \cap mR] = \alpha S + Ann_S(m)$.

Proof: First we show

$$\alpha S + Ann_S(m) \subseteq Ann_S[Ker\alpha \cap mR]$$

Let $x \in Ker\alpha \cap mR$

$\Rightarrow x \in Ker\alpha$ and $x \in mR$

$\Rightarrow \alpha x = 0$ and $x = mr$, $r \in R$

Let $f \in Ann_S(m) \Rightarrow mf = 0 \Rightarrow r(mf) = 0$
 $\Rightarrow (mr)f = 0 \Rightarrow xf = 0$

Then $x[\alpha S + Ann_S(m)] = x\alpha S + x Ann_S(m)$
 $= \alpha x S + x Ann_S(m)$
 $= 0 + 0 = 0$

Therefore, $\alpha S + Ann_S(m) \subseteq Ann_S[Ker\alpha \cap mR]$

Next, $Ann_S[Ker\alpha \cap mR] \subseteq \alpha S + Ann_S(m)$

Let $\beta \in Ann_S[Ker\alpha \cap mR]$

We claim that $Ann_R(\alpha m) = Ann_R(\beta m)$

Let $x \in Ann_R(\alpha m) \Rightarrow x(\alpha m) = 0$
 $\Rightarrow \alpha(xm) = 0 \Rightarrow \alpha(mx) = 0$
 $\Rightarrow mx \in Ker\alpha \cap mR$

Therefore $(mx)\beta = 0 \Rightarrow (xm)\beta = 0$
 $\Rightarrow x\beta(m) = 0$
 $\Rightarrow x \in Ann_R(\beta m)$

Therefore, $Ann_R(\alpha m) \subseteq Ann_R(\beta m)$

Similarly $Ann_R(\beta m) \subseteq Ann_R(\alpha m)$

Therefore, $Ann_R(\alpha m) = Ann_R(\beta m)$
 $\Rightarrow \alpha m S = \beta m S$

$$\begin{aligned} \text{Then } \exists \gamma \in S \text{ such that } (\alpha \circ \gamma)m &= \beta m \\ \Rightarrow (\beta - \alpha \circ \gamma)m &= 0 \\ \Rightarrow \beta - \alpha \circ \gamma &\in \text{Ann}_S(m) \end{aligned}$$

So $\beta \in \alpha S + \text{Ann}_S(m)$

Thus $Ann_S[Ker \alpha \cap mR] = \alpha S + Ann_S(m)$.

Theorem 3.5: For a uniform module M the following conditions are equivalent:

- (i) M is essential pseudo p – injective.
- (ii) M is pseudo p – injective.

Proof: (i) \Rightarrow (ii):

Let M be an essential pseudo p-injective module.

We are to show that M is pseudo p - injective.

Let $f, g: mR \rightarrow M$ be monomorphisms where $mR \in \mathcal{F}$.

Given M is a uniform module, therefore all submodules of M are essential in M . So by the definition of essential pseudo p -injectivity of M , there exists $h \in \text{End}(M)$ such that $f = h \circ g$.

Thus, M is pseudo p – injective.

Next (ii) \Rightarrow (i): Follows from definition.

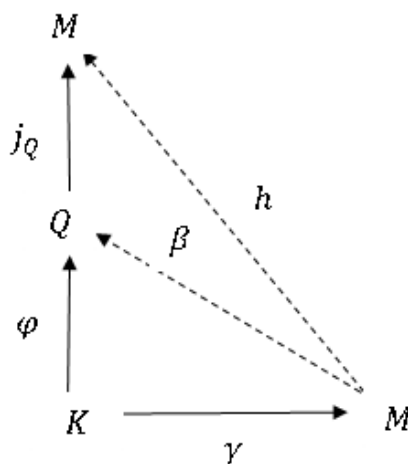
Note: If M has no essential extension then M is essential pseudo p -injective.

Theorem 3.6: Let M be an essential pseudo p –injective module then M has no proper essential submodule $K \in \mathcal{F}$ where K is isomorphic to a direct summand of M .

Proof: Let Q be a direct summand of M and K be any essential submodule of M such that $K \cong Q$.

Let $\varphi: K \rightarrow Q$ be the isomorphism, $\gamma: K \rightarrow M$ be the natural map, $j_Q: Q \rightarrow M$ be the injection map and $\pi_Q: M \rightarrow Q$ be the projection mapping.

Since M is essential pseudo p – injective then there exists $h \in \text{End}(M)$ such that the following diagram commutes.



Thus $h \circ \gamma = j_0 \circ \varphi$.

Again define $\beta : M \rightarrow Q$ by $\beta = \pi_0 \circ h$

And $\gamma' : M \rightarrow K$ by $\gamma(m) = \varphi^{-1} \circ \beta(m)$

Again, let $y \in \text{Im}(h \circ \varphi)$
 $\Rightarrow (h \circ \varphi)(y) = m$
 $\Rightarrow (f \circ \varphi^*)(y) = m$
 $\Rightarrow y \in \text{Im}(f \circ \varphi^*)$
 $\Rightarrow y \in \text{Im } \varphi$

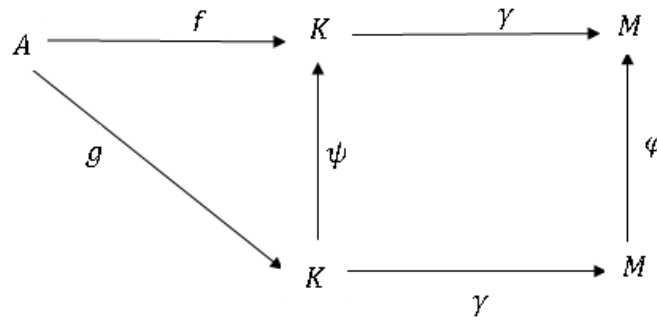
Therefore $\text{Im}(h \circ \varphi) \subseteq \text{Im } \varphi$
 $\Rightarrow h(\text{Im } \varphi) \subseteq \text{Im } \varphi$.

Thus, $\text{Im } \varphi$ is invariant under mono – endomorphism of M .

Theorem 3.8: Let M be an essential pseudo p –injective module and $K \in \mathcal{F}$ be a submodule of M , then K is essential pseudo p –injective if K is stable under mono endomorphism of M .

Proof: Let $\gamma: K \rightarrow M$ be the natural map $f, g: mR \rightarrow K$ be a monomorphisms.

Then by essential pseudo p – injectivity of M , there exists $\varphi \in \text{End}(M)$ such that the following diagram commutes.



Thus $\gamma \circ f = \varphi \circ \gamma \circ g$.

We define $\psi: K \rightarrow K$ as $\psi(x) = \varphi(x)$, $x \in K$.

Then, ψ is well defined as let $x_1 = x_2 \Rightarrow \varphi(x_1) = \varphi(x_2) \Rightarrow \psi(x_1) = \psi(x_2)$.

Thus ψ is well defined and

$$\gamma \circ \psi = \varphi \circ \gamma \Rightarrow \gamma \circ \psi \circ g = \varphi \circ \gamma \circ g \Rightarrow \gamma \circ \psi \circ g = \gamma \circ f$$

$\Rightarrow \psi \circ g = f$, since γ is natural map.

Hence, K is essential pseudo p – injective.

Theorem 3.9: Let M be an essential pseudo p –injective module and $g: N \rightarrow M$ be any monomorphism, where $N \in \mathcal{F}$. Then N is essential pseudo p – injective.

Proof: Given, M is aessential pseudo p –injective module and $g: N \rightarrow M$ be any monomorphism. Then by theorem 3.7, $\text{Im } g = \text{Im}(h \circ g)$ is stable under h .

But $\text{Im}(g)$ is a submodule of M .

Then by theorem 3.8, we have $\text{Im } g$ is essential pseudo p –injective module.

Since $\text{Im } g \cong N$ and $\text{Im } g$ is essential pseudo p – injective, therefore N is essential pseudo p – injective module.

Theorem 3.10: Let M be an essential pseudo p –injective module and $N \in \mathcal{F}$ be submodule of M stable under mono-endomorphism of M , then N is essential pseudo stable.

Proof: Given, M is an essential pseudo p –injective module and N is an essential cyclic submodule of M stable under mono–endomorphism of M , then by theorem 3.8, N is essential pseudo p –injective.

Also as N is stable under mono-endomorphism of M , we have $h(N) \subseteq N$, where $h \in \text{End}(M)$. Thus N is essential pseudo stable.

Theorem 3.11: Let M be an essential pseudo p -injective module and T be an essential pseudo stable sub module of M , then T is essential pseudo p -injective.

Proof: Let $g, f: mR \rightarrow T$ be any monomorphisms and $\gamma: T \rightarrow M$ be the natural map.

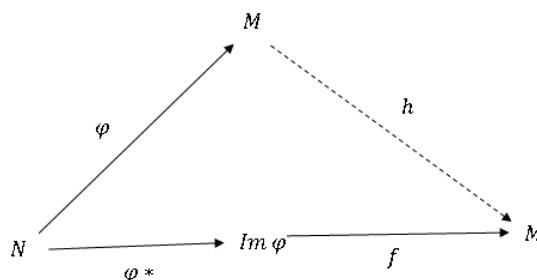
Then by essential pseudo p -injectivity of M , there exists $h \in \text{End}(M)$ such that $\gamma \circ f = h \circ \gamma \circ g$.

Since T is essential pseudo stable, therefore $h(T) \subseteq T$.

Hence by theorem 3.8, T is essential pseudo p – injective.

Theorem 3.12: If M is an essential pseudo p -injective module and $\varphi: mR \rightarrow M$ is any monomorphism, then $Im \varphi = K$ is an essential pseudo stable submodule of M .

Proof: Here $\varphi: mR \rightarrow M$ induces an isomorphism $\varphi*: mR \rightarrow Im\varphi$.



Let $f: Im\varphi \rightarrow M$ be the natural map. Since, M is essential pseudo p -injective module therefore there exists $h \in End(M)$ such that the following diagram commutes.

Thus $f \circ \varphi * = h \circ \varphi$.

Now, let $h(K) \not\subseteq K$. Then there exists $k \in K$ such that $h(k) \in h(K)$ and $h(k) \notin K$.

Then $0 \neq h(k) = h(\varphi(k')) = (h \circ \varphi)(k') = (f \circ \varphi*)(k') \in \text{Im}(f \circ \varphi*)$ as $K \subseteq \text{Im } f \circ \varphi*$, which is a contradiction.

Thus, $h(K) \subseteq K$

$\Rightarrow K$ is essential pseudo stable.

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