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REVERSE VERTEX- MAGIC LABELING OF GENERALIZED PETERSEN GRAPHS AND CONVEX POLYTOPES

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ABSTRACT

***A** reverse vertex – magic labeling on a graph with v vertices and e edges is a one- to- one map taking the vertices and edges on to the integers 1, 2, 3,…… $v+e$ with the property that subtract the label on the vertex and the labels of its incident edges is constant, independent of the choice of the vertex. We give the reverse vertex –magic labeling for Generalized Petersen Graphs and Convex Polytopes.*

1. INTRODUCTION

All graphs considered in this paper are finite , simple and undirected. The graph G has vertex set V = V(G) and edge set E = E(G) and we let e = $|E|$ and $v=|V|$. A general reference for graph theoretic notations in [9]. In this paper, we will deal only with connected graphs, although the concepts apply equally to graphs with more than one connected component.

A labeling for a graph is a map that takes graph elements to numbers. In this paper the domain is the set of all edges, giving rise to labeling.

In [8] we introduced the notion of a reverse super vertex labeling. This is an assignment of the integers from 1 to v+ e to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex subtract to a fixed constant. More formally, a one – to- one map λ from E∪V on to the integers {1, 2, 3… v+ e} is a reverse vertex – magic labeling if there is a constant k so that for every vertex x ,

$$f(x)-\sum f(xy)=k \tag{1}$$

Where the sum is over all vertices y adjacent to x.

Set M = v+ e and let s_v be the sum of the vertex labels and s_e, the sum of the edge labels. Clearly since the labels are the numbers 1, 2, …., M, we have as the sum of all labels

$$s_v+s_e=\sum_{1}^{M} i=\binom{M+1}{2}$$

At each vertex x_i we have

$$f(x)-\sum f(xy)=k\ .$$

We sum this over all v vertices x_i.This adds each vertex label twice so that $s_v-2s_e=vk$ (2)

Combining these two equalities gives us $\binom{M+1}{2}-3s_e=vk$ (3)

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The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the e smallest labels or at the other extreme, the e largest labels or any thing between consequently we have

$$\sum_{1}^{e} i \le S_e \le \sum_{v+1}^{M} i \qquad (4)$$

A singular results hold for S_e. Combining (3) & (4), we get

$$\binom{e+1}{2} \le \frac{1}{3}\left\{\binom{M+1}{2} - vk\right\} \le \binom{M+1}{2} - \binom{v+1}{2}$$

$$3\binom{e+1}{2} \le \left\{\binom{M+1}{2} - vk\right\} \le 3\binom{M+1}{2} - 3\binom{v+1}{2}$$

$$-3\binom{M+1}{2} + 3\binom{v+1}{2} \le vk - \binom{M+1}{2} \le -3\binom{e+1}{2}$$

$$3\binom{v+1}{2} - 2\binom{M+1}{2} \le vk \le \binom{M+1}{2} - 3\binom{e+1}{2} \qquad (5)$$

Which will give the range of the feasible values for k.

It is clear from the definition of reverse vertex-magic labeling that when k is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. In this paper, we give new reverse vertex –magic labeling of several classes of graphs.

2. LABELINGS OF GENERALIZED PETERSEN GRAPHS

Suppose g is a reverse edge-labeling of a graph, that is, a one-to-one map from E onto the integers {1, 2,…, e}. Then the weight w(x) of a vertex $x \in$ V is defined as the sum of labels assigned to all edges incident to *x*.

An edge labeling g is said to be consecutive if the weights of all vertices constitute a set of consecutive integers. Two edge – labeling g and g' are said to be complementary if there is a constant c such that $w_g(x) + w_{g'}(x) = c$ for all x $\in$ V.

Let I = {1, 2,…, n} be an index set. For simplicity we use the convention that $x_{j,n+1} = x_j$, for j = 1, 2,…,6.

A generalized Petersen graph P(n, m), $1 \le m \le \frac{n}{2}$, consists of an outer n-cycle $y_1, y_2, \ldots y_n$, a set of n spokes $y_i x_i$, $1 \le i \le n$, and n inner edges $x_i x_{i+m}$, $1 \le i \le n$, with indices taken modulo n. The standard Petersen graph is the instance P(5,2).

P(n,m) is regular of degree 3 and has v = 2n vertices and e = 3n edges; thus M = 5n.

Using (5), we can readily determine the feasible values of k for the generalized Petersen graphs P(n, m):

$$-\frac{19n}{2} - 1 \le k \le -\frac{n}{2} - 1$$

It was shown in [1] that for n≥4, n even and $1 \le m \le \frac{n}{2} - 1$ the generalized Petersen graph P(n, m) has a consecutive reverse edge-magic labeling defined by the bijective mapping g_1: E[P(n, m)]→{1,2,….,3n}.

- $g(y_i y_{i+1}) = \frac{5n+i+1}{2}\delta(i) + \frac{n-i+2}{2}$
- $g_1(y_i x_i) = \frac{3n-i+1}{2}\delta(i) + \left[\left(n + \frac{i}{2} - 2\right)\rho(i,4) + \left(\frac{n+i}{2} - 2\right)\rho(6,i)\right]\delta(i+1)$
- $g_1(x_i x_{i+m}) = \frac{3n+i+1}{2}\delta(i) + \left[\left(2n - \frac{i}{2} + 3\right)\rho(i,4) + \left(\frac{5n-i}{2} + 3\right)\rho(6,i)\right]\delta(i+1)$

$*$ for i∈I where $\delta(x)=\begin{cases}0 & if\ x\equiv 0(\bmod 2)\\ 1 & if\ x\equiv 1(\bmod 2)\end{cases}$

$$\rho(x,y)=\begin{cases}0 & if\ x>y\\ 1 & if\ x\leq y\end{cases}$$

The weight of vertices under the mapping g_1 constitute the sets

$W_1=\left\{w_{g_1}(y_i):i\in I\right\}=\left\{\frac{7n}{2}+2,\frac{7n}{2}+3,\ldots..,\frac{9n}{2}+1\right\}$ and

$W_2=\left\{w_{g_2}(x_i):i\in I\right\}=\left\{\frac{9n}{2}+2,\frac{9n}{2}+3,\ldots..,\frac{11n}{2}+1\right\}$

Label the edges of the generalized Petersen graph P(n, m) by the consecutive labeling g_1. If g_2 is the complementary vertex labeling with values in the set $\left\{|E|+1,|E|+2,\ldots..,|E|+|V|\right\}=\left\{3n+1,3n+2,\ldots..,5n\right\}$. then the labelings g_1 and g_2 combine to give a reverse vertex – magic labeling of P(n, m). Since the largest labels are assigned to the vertices, it is easily seen that the resulting magic constant $k=-\frac{n}{2}-1$ is the largest possible.

Thus we have the following

Theorem 1: For n≥4, n even and $1\leq m\leq\frac{n}{2}-1$, the generalized Petersen graph P(n, m) has a reverse vertex – magic labeling with $k=-\frac{n}{2}-1$.

We know that if a regular graph G possesses a reverse vertex – magic total labeling λ with magic constant k, then G also has a dual labeling λ' having magic constant k' = (1-r)(M+1)-k, where r is the number of edges incident to the vertex. Since P(n,m) is regular, it has a dual labeling. Hence

Corollary1: For n ≥ 4, n even and $1\leq m\leq\frac{n}{2}-1$, the generalized Petersen graph P(n, m) has a reverse vertex –magic labeling with $k=-\frac{19n}{2}-1$.

Proof: In this case the dual labeling g' is defined by

- $g_1'(u)=|E|+|V|+1-g_1(u)$ for any edge u∈E(P(n. m)),
- $g_2'(x)=|E|+|V|+1-g_2(x)$ for any vertex x∈V(P(n, m)).

Since the magic constant for g is $k=-\frac{n}{2}-1$, then g' is a reverse vertex-magic labeling with dual magic $k'=-\frac{19n}{2}-1$.

Definition 2: The Prism D_n, n≥3, is a trivalent graph which can be defined as the Cartesian product P_2xC_n of a path on two vertices with a cycle on n vertices. We note that the prism D_n is the generalized Petersen graph P(n,1).

Corollary 2: For n ≥ 4, n even, the Prism D_n has a reverse vertex –magic labeling with $k=-\frac{n}{2}-1$ and another one with $k=-\frac{19n}{2}-1$.

3. LABELINGS OF SOME FAMILIES OF CONVEX POLYTOPES

In this section, we shall investigate the graphs of two families of convex polytopes. First we consider the graphs R_n consisting of 2n 5-sided faces, n 6-sided faces and a pair of n-sided faces, embedded in the plane and labeled as in fig 2. Using equation (5) where v=6n and e=9n, we can determine the feasible values of the magic constant k for the graph $R_n = -\frac{57n}{2} - 1 \le k \le -\frac{3n}{2} - 1.$

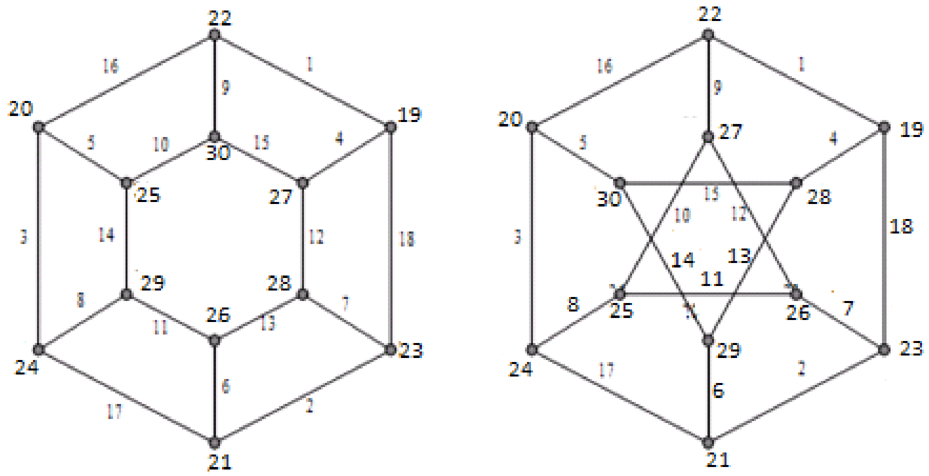

Figure 1: Labeling of $P(6;1)$ and $P(6;2)$.

The dual graph of R_n is a planar graph B_n which has been investigated in [2]. There it was shown that B_n has a face anti-magic labeling g ie an ed ge labeling in which the sum around each face is a constant. To obtain a reverse vertex – magic labeling of R_n, we make use of the labeling $h_1 = E(R_n) \to \{1,2,3,....,9n\}$ which is a modification of the labeling g from [2]. It is defined as follows (with δ and ρ as defined in [6] and [7]):

- $h_1(x_{1,i}x_{1,i+1}) = \left[(8n+i)\delta(i) + (n-i)\delta(i+1)\right]\rho(i,n-1) + n\rho(n,i)$
- $h_1(x_{1,i}x_{2,i}) = \left(\frac{5n}{2} + i\right)\delta(i) + \left(\frac{5n}{2} - i + 1\right)\delta(i+1)$
- $h_1(x_{2,i}x_{3,i}) = \left(\frac{15n}{2} - i + 1\right)\delta(i) + \left(\frac{9n}{2} - i + 1\right)\delta(i+1)$
- $h_1(x_{3,i}x_{2,i+1}) = \left(\frac{13n}{2} + i\right)\delta(i) + \left(\frac{9n}{2} - i + 1\right)\delta(i+1)$
- $h_1(x_{3,i}x_{4,i}) = (n+1)\rho(i,1) + \left[\frac{3n-i+3}{2}\delta(i) + \left(\frac{9n}{2} + i\right)\delta(i+1)\right]\rho(2,i)$
- $h_1(x_{4,i}x_{5,i}) = \frac{11n+i+1}{2}\delta(i) + \left(\frac{11n}{2} - i + 1\right)\delta(i+1)$
- $h_1(x_{5,i}x_{4,i+1}) = \frac{13n-i+1}{2}\delta(i) + \frac{5n+i}{2}\delta(i+1)$

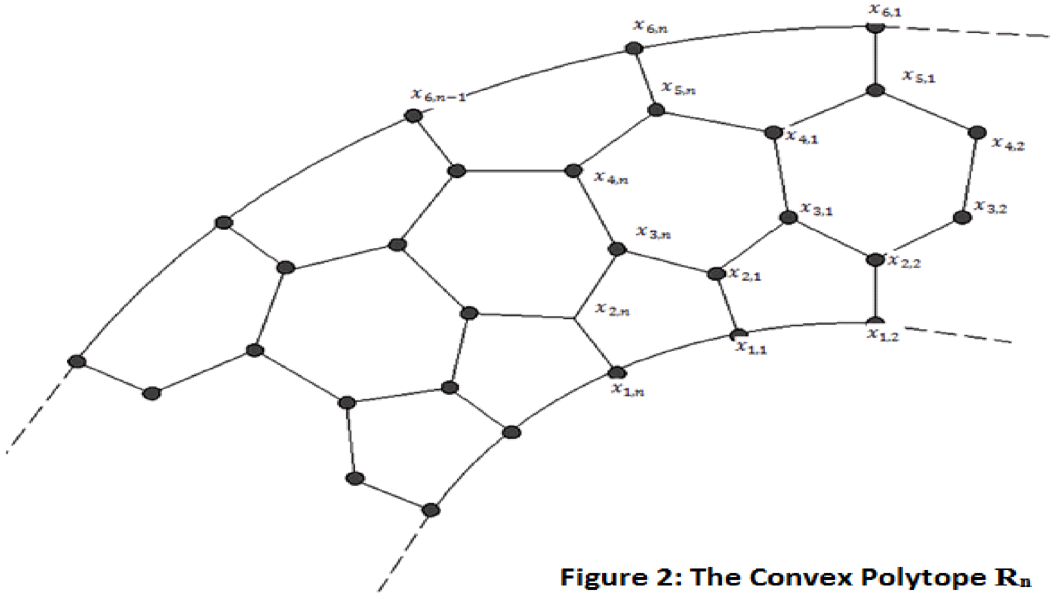

Figure 2: The Convex Polytope R_n

- $h_1(x_{5,i}x_{6,i}) = \left(\frac{5n}{2} - i + 1\right)\delta(i) + \left(\frac{5n}{2} + i\right)\delta(i+1)$
- $h_1(x_{6,i}x_{6,i+1}) = (8n+i+1)\delta(i) + (n-i+1)\delta(i+1)$
- for i∈I.

The edge labeling h_1 is consecutive: the weights of vertices in turn assume the values $\frac{21n}{2}+2, \frac{21n}{2}+3, ..., \frac{33n}{2}+1$.

If h_2 is the complimentary vertex labeling with values in the set $\{|E(R_n|+1, |E(R_n)|+2, ..., (|E(R_n)|+|V(R_n|\}$ = $\{9n+1, 9n+2, ..., 15n\}$ then the labeling h_1 and h_2 combine to give a reverse vertex- magic labeling of R_n with the magic constant $k = -\frac{3n}{2} - 1$.

Theorem 2: For n≥4, n even, the plane graph R_n has a reverse vertex- magic labeling with $k = -\frac{3n}{2} - 1$.

Since R_n is regular, we have a dual labeling as before.

Corollary 3: For n≥4, n even, the plane graph R_n has a reverse vertex- magic labeling with $k = -\frac{57n}{2} - 1$.

The final graphs we investigate are the antiprisms A_n, n≥3, a family of planar graphs that are regular degree 4. These are Archimedean convex polytopes and, in particular, A_n is the octahedron.

We will denote the convex set of A_n by V = {x_i: i∈I} ∪{y_i: i∈I} and the edge set by $E = \{(x_i, x_{i+1}) : i \in I\} \cup \{(y_i, y_{i+1}) : i \in I\} \cup \{(x_i, y_i) : i \in I\} \cup \{(y_i, x_{i+1}) : i \in I\}$ as in figure 3.

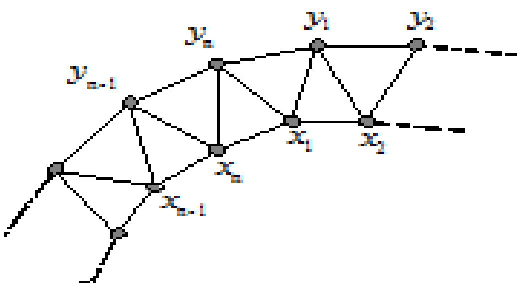

Figure 3: The antiprism A_n.

From (5) we get the range of feasible values for k:

$$-15n-\frac{3}{2}\le k\le -3n-\frac{3}{2}$$

Theorem 3: For n≥ 4, n even, the plane graph A_n has a reverse vertex- magic labeling with $k=-9n$.

Proof: We construct an edge labeling f_1 of A_n, n = 2m, m ≥2, in the following way:

- $f_1(x_ix_{i+1})=6n\rho(i,1)+[(5n+i-1)\delta(i)+i\delta(i+1)]\rho(2,i)$
- $f_1(y_iy_{i+1})=[(5n+i)\delta(i)+(2n+i+1)\delta(i+1)]\rho(i,n-1)+(2n+1)\rho(n,i)$
- $$\begin{aligned}f_1(x_iy_i)&=(3n+1)\alpha(1,i,1)+(5n-2i+3)\alpha(2,i,m+1)+(3n+3)\alpha(m+2,i,m+2)\\&\quad+(5n-2i+3)\alpha(m+3,i,n-1)+(4n-1)\alpha(n,i,n)\end{aligned}$$

.

- $f_1(y_ix_{i+1})=2n-2i+1$ Here $\alpha(x,y,z)=\begin{cases}1 & if\ x\le y\le z\\ 0 & otherwise\end{cases}$

The edge labeling f_1 is a one-to-one map from E(An)on to the set{i:i∈I} ∪ {n+2j-1: j= 1,2,…,2n} ∪ {5n+i, i∈I}. The weights of the vertices under the edge labeling f_1 constitute the set

$$w=\left\{w_{f_i}(x):x\in V(A_n)\right\}=\{10n+2j;\ j=1,2,3,....,2n\}$$

If f_2 is the complementary vertex labeling with values in the set {n+2j: j = 1, 2,…, 2n} then the labeling f_1 and f_2 combine to give reverse vertex- magic labeling of A_n with the magic constant $k=-9n$.

Theorem 4: For n≥ 4, n even, the antiprism A_n has a reverse vertex – magic labeling with $k=-9n$.

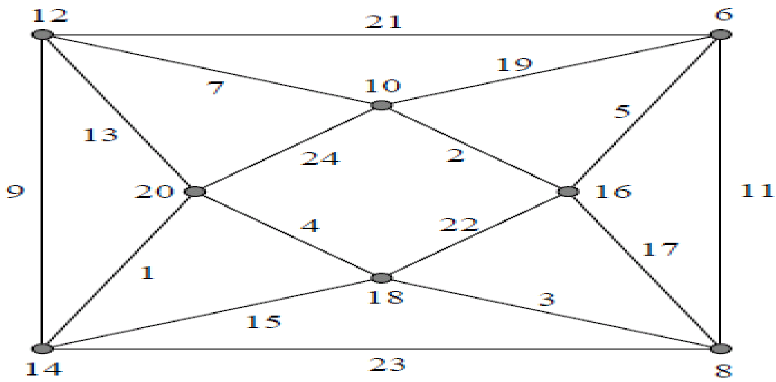

Figure 4: Labeling of the antiprism A_4.

4. OPEN PROBLEMS

We have shown that there exist reverse vertex – magic labeling for the generalized Petersen graph P(n, m) for n ≥ 4, n even. We conjecture that

Conjecture 1: There is a reverse vertex-magic labeling for the Prism D_n = P(n,1) for all n ≥ 3
More strongly,

Conjecture 2: There is a reverse vertex-magic labeling for the generalized Petersen graph P(n, m) for all n ≥ 3 and $2\le m\le\frac{n}{2}$ . .

We have not yet found a construction that will produce a reverse vertex- magic labeling for the plane graph R_n for n odd. However, we suggest the following

Conjecture 3: There is a reverse vertex-magic labeling for the plane graph R_n for all n≥3.

Open Problem1: Find a reverse vertex- magic labeling for the antiprism A_n for all odd n≥3.

The Ladder L_n , n≥3 , can be viewed as the Cartesian product $P_2 \times P_n$ of a path on two vertices and a path on n vertices , or as a Prism D_n with two edges deleted.

Open Problem2: Find a reverse vertex- magic labeling for the Ladder L_n.

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