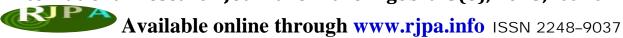
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# **ON ECM-P-INJECTIVE MODULES**

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#### **ABSTRACT**

**T**he aim this article is to explore the characterization of M-cyclic submodule. Let R be a ring. M and N are R-modules. A module N is called ECM-principally injective module (briefly, ECM-P-injective) if every R-homomorphism from essentially M-cyclic submodule of M to N, can be extended to M. In this paper we obtain to investigate some characteristics of M-principally injective module. Using the notion EC-M-cyclic submodule of M.

**Key words:** EC-M-cyclic, M-principally injective, ECM-principally injective, pseudo M-principally injective and pseudo quasi-principally injective.

#### 1. INTRODUCTION

Through this paper, by a ring R we always mean as associative with identity and every R-module is unitary. The notion principally injective module was introduced by Camollo [9]. Nicholson, park and Yousif studied the structure of principally injective and Quasi- principally injective modules [10]. N is called M- principally injective if Tansee and Wongwai also extended this notion. A right R-module every R-homomorphism from an M-cyclic submodule of M to N, can be extended to M. A module M is called Quasi- principally injective if it is M- principally injective. A submodule K of M is called essential submodule if  $K \cap L \neq 0$  for every nonzero submodule L of M. In other words  $K \cap N = 0 \Rightarrow K = 0$  (briefly;  $K \leq^e M$ ). In this case M is called essential extension of K. A monomorphism  $f: K \to M$  is said to be essential if  $\inf \leq^e M$ . A submodule K is called M-cyclic if K is image of element of S. ( $S = \operatorname{End}_R(M)$ ) denotes endomorphism ring of M). A submodule K is called essentially M-cyclic (briefly; EC-M-cyclic) if it is the image of element of S and it's inclusion map is essential.

#### 2. PRELIMINARY RESULTS

In this section, we study of essential submodule.

**Lemma.2.1:** Let M, N be right R-modules and let f:  $N \to M$  be a homomorphism, if M' is an essential sub module of N, then  $f^1(M')$  is essential sub module of N.

Proposition 2.1: Let K and N be sub modules of an R-module M. Then

- (i)  $K \le^e M \Leftrightarrow K \le^e N \text{ and } N \le^e M$ .
- $(ii) \quad K \leq^e M \Leftrightarrow K \cap Rm \neq 0 \quad \forall \ 0 \neq m \in M.$
- (iii) Given  $K \subset N$  if  $N/K \le^e M/K$  then  $N \le^e M$ .
- (iv)  $K \cap N \leq^e M \Leftrightarrow K \leq^e M$  and  $N \leq^e M$ .
- (v) If  $K \leq^e M$  then  $K \cap N \leq^e M$ .
- (vi)  $K \leq^e M \Leftrightarrow$  for each  $0 \neq m \in M \exists$  an  $r \in R$  such that  $0 \neq mr \in K$ .
- (vii)  $K_1 \oplus K_2 \leq^e M_1 \oplus M_2 \Leftrightarrow K_1 \leq^e M_1$  and  $K_2 \leq^e M_2$  for each  $K_1 \leq M_1 \leq M$  and  $K_2 \leq M_2 \leq M$ .
- (viii) If  $M = \bigoplus_{i=1}^{n} M_i$  and  $K_i \leq M_i$  for each  $i \in I$ , then  $= \bigoplus_{i=1}^{n} N_i \leq^e M$ .

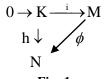
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#### Proof:

- i) Given that  $K \leq N$  and  $K \cap L = 0$ ,  $K \leq M$ . Clearly  $L \leq N \leq M \Rightarrow L \leq M$ . Since  $K \leq^e M$ , then  $K \cap L = 0 \Rightarrow L = 0$  implies  $K \leq^e N$ . Let  $T \leq M$  such that  $N \cap T = 0$ ,  $K \cap T \leq N \cap T = 0$ 
  - $\Rightarrow$  N  $\cap$  T = 0, since K  $\leq$ e M, then T = 0  $\Rightarrow$  N  $\leq$ e M. Conversely, Let K  $\cap$  S = 0 for some S  $\leq$  M
  - $\Rightarrow$  K  $\cap$  (S  $\cap$  N) = 0. Since K  $\leq$ <sup>e</sup> N and S  $\cap$  N  $\leq$  N, S  $\cap$  N = 0 with N  $\leq$ <sup>e</sup> M  $\Rightarrow$  S = 0 so K  $\leq$ <sup>e</sup> M.
- ii) We have  $N \cap K \leq N \leq M$ , Since  $N \cap K \leq^e M \Rightarrow N \leq^e M$ . Similarly  $N \cap K \leq N \leq M$ , Since  $N \cap K \leq^e M \Rightarrow K \leq^e M$ . Similarly  $N \leq^e M$ ,  $K \leq^e M$ . To prove  $N \cap K \leq^e M$ . Let  $(N \cap K) \cap T = 0$  for some  $T \leq M \Rightarrow N \cap (K \cap T) = 0$ , since  $N \leq^e M \Rightarrow K \cap T = 0$  and also  $K \leq^e M$ , therefore T = 0, by (i) we get  $K \cap N \leq^e M$ .
- iii) and iv) [1].
- $v). \ \text{Let} \ (K \cap N) \cap T = 0 \ \text{ for some} \ T \leq K \leq M, \ \text{therefore} \ K \cap (\ N \cap T) = 0, \ \text{since} \ K \leq^e M \ \text{and} \ N \cap T \leq 0 \Rightarrow N \cap T = 0 \ \text{and} \ N \cap T = T \Rightarrow T = 0, \ \text{Hence} \ K \cap N \leq^e N.$
- vii) and viii) [1]. //

#### 3. ECM-P-INJECTIVE MODULE

**Definition 3.1:** An R-module N is called essential M-principally injective module (ECM-P-injective), if every R-homomorphism from EC-M-cyclic submodule K of M to N, can be extended to M, in general the following diagram is commutative,



i.e.  $\Phi.i = h$ . where  $\phi \in End_R(M)$  and  $K = \phi(M) \leq^e M$ .

# Example 3.1:

- (i) Z is essential sub module of the Z-module Q, is cyclic, but not Q-cyclic, for every non zero homomorphism  $f: Q \to Q$  is an epimorphism.
- (ii) Let  $M = Z_1 \oplus Z_2 \oplus Z_3$  is a z-module, since  $M/Z_3 = Z_2 \oplus Z_2$ , then  $Z_2 \oplus Z_2$  is EC-M-cyclic, but  $Z_2 \oplus Z_2$  is not cyclic.

(M-cyclic submodule and cyclic module both are completely different concepts)

**Lemma 3.1:** Let M and N be R-modules. Then N is ECM-P-injective if and only if for each  $s \in S = End_R(M)$ .  $Hom_{_{\mathbf{P}}}(M,N)_{_{\mathbf{S}}} = \left\{ f : Hom_{_{\mathbf{P}}}(M,N) : f(kers) = 0 \right\}$ 

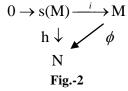
**Proof:** Assume that N is ECM-P-injective module. We want to show that

$$\text{Hom}_{R}(M, N)_{S} = \{f : \text{Hom}_{R}(M, N) : f(\text{kers}) = 0\}$$

It is clear that

$$\operatorname{Hom}_{R}(M, N)_{S} \subseteq \{f : \operatorname{Hom}_{R}(M, N) : f(\ker S) = 0\}$$

Let  $f \in Hom_R(M, N)$  such that  $f(kers) = 0 \Rightarrow kers \subset kerf$ . Then there is an homomorphism  $i: s(M) \to M$  such that i.s = f. Since N is ECM-P-injective module.



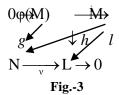
There exists an R-homomorphism  $\phi: M \to N$  such that  $\phi.i = h$  where the inclusion map  $i: s(M) \to M$  is essential monomorphism with s(M) is large M-cyclic submodule of M. Then  $\phi s \in Hom_R(M,N)$  and kers  $\in ker\phi s \Rightarrow \phi s(kers) = 0$ . By assumption  $\phi s(M) = u[s(M)] = u[i(s(M))] \Rightarrow us(M)$  is also large M-cyclic submodule of M. This show that N is ECM-P-injective module.//

**Theorem 3.1:** Let M and N be R-modules. Then M is N-Principally projective module and every EC-M-cyclic submodule of N is ECM-P-injective if and only if Nis ECM-P-injective module and EC-M-cyclic submodule of M is ECM-P-injective.

**Proof:** Let M be N-Principally projective module and suppose that every EC-M-cyclic submodule of N is ECM-P-injective. Since n is trivially M-cyclic, so N is ECM-P-injective. Let  $\phi \in End_{\mathbb{R}}(M)$ .

Let  $v: M \to L$  be small epimorphism and let  $h: \phi(M) \to L$  be any homomorphism, where  $\phi(M)$  is EC-M-cyclic submodule of M.

Consider the diagram:



Where  $i:\phi(M)\to M$  is an inclusion monomorphism, implies  $\phi(M)\leq^e M$  we have L is M-cyclic i.e. L is ECM-injective. There exists an epimorphism  $1:M\to L$  such that 1.i=h and the sequence  $0\to\phi(M)\stackrel{i}{\longrightarrow} M\stackrel{l}{\longrightarrow} L\to 0$  is exact. Since M is N-projective module this implies, so there exists an homomorphism  $t:M\to N$  such that v.t=1 and the map  $g:\phi(M)\to N$  such that g=t.i.

Now  $v \cdot g = v \cdot t \cdot i = 1 \cdot i = h$ . This shows that every M-cyclic sub module of M is N-P projective.

Conversely, suppose that every M-cyclic sub module of M is N-P projective and N is ECM-P-injective.

Consider the diagram:

where  $i:\phi(M)\to M$  is inclusion monomorphism and  $h:\phi(M)\to B$  is any homomorphism,  $g:M\to B$  is an required small epimorphism. Since  $\phi(M)$  is N-projective module, thus there exists a homomorphism  $g:\phi(M)\to N$  such that v.g=h. But N is ECM-P-injective, so there is an homomorphism  $t:M\to N$  such that t.i=g, Define  $1:M\to N$  by 1=v.t. Now 1.i=v.t.i=v.g=h.

**Theorem 3.2:** The following are equivalent for a projective module M.

- (i) Every small M-cyclic sub module of M is projective.
- (ii) Every factor module of an ECM-P-injective is ECM-P-injective.
- (iii) Every factor module of an injective R-module is ECM-P-injective.

### **Proof:**

(i)  $\Rightarrow$  (ii): Let N be an ECM-injective module, X is small M-cyclic sub module of N. let  $s \in End_R(M)$ . Consider the diagram:

Let  $\phi: s(M) \to N/X$  be an R-homomorphism by (i) there exists an R-homomorphism  $\hat{\phi}: s(M) \to N$  such that  $\phi = \eta.\hat{\phi}$ . Where  $\eta: N \to N/X$  is the natural epimorphism. Since N is ECM-P-injective, there exists an R-homomorphism  $t: M \to N$ , which is essential extension of  $\hat{\phi}$  to M. Then  $\mu.t$  is essential extension of to M i.e. factor module is ECM-P-injective.

(ii) ⇒(iii): Clear.

(iii)  $\Rightarrow$  (i): Let s(M) be an small M-cyclic sub module of M and  $h:A\to B$  is an epimorphism and let  $\alpha\colon s(M)\to B$  be an homomorphism imbed A in an injective module E.  $B\cong A/kerh$  is a submodule of LMP-injective module E/kerh. Let a map  $\alpha\colon s(M)\to E/kerh$  by hypothesis we can extend  $\hat{\alpha}\colon M\to E/kerh$ . Since M is projective,  $\hat{\alpha}$  can be lifted to  $g:M\to E$  such that  $\eta.g=\hat{\alpha}$  where  $\eta$  is natural map. It is clear that  $g(s(M))\subset A$ . Therefore we have lifted  $\alpha$ , Implies every small M-cyclic sub module of is projective.//

# 4. EC- PSEUDO QUASI- PRINCIPALLY INJECTIVE MODULE (EC-PQ-P-INJECTIVE MODULE)

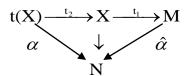
**Definition 4.1:** Let M be right R-module. A right R-module N is called essentially pseudo-M-principally injective module (EC-PM-P- injective) if every R- monomorphism from EC-M-cyclic submodule of M to N can be extended to an  $\operatorname{End}_{\mathbb{R}}(M)$ . The module M is called essentially pseudo Quasi- principally injective module.

**Lemma 4.1:** Every EC-X-cyclic submodule of X is an EC-M-cyclic submodule of M for every EC-M-cyclic submodule X of M.

**Proof:** [11].

**Proposition 4.1:** N is EC-PM-P-injective if and only if N is EC-PX-P-injective foe every EC-M-cyclic sub module of M.

**Proof:**  $\Rightarrow$  Let X = s(M) is an EC-M-cyclic sub module of M. t(X) is a EC-X-cyclic submodule of X and let  $\phi: t(X) \to N$  be an R-essential monomorphism. Since  $t, s \in S$  and t(M) = t(X). Since N is EC-PM-P-injective, there exists an R-homomorphism  $\hat{\alpha}: M \to N$  such that  $\alpha = \hat{\alpha}.t_1.t_2$ 



Where  $t_1: X \to M$ ,  $t_2: t(X) \to X$  both are inclusion monomorphisms. Then  $\hat{\alpha}.t_2$  is the extension of  $\alpha$ . [5. pro. 5.12] N is EC-PM-P-injective.

it is clear.//

**Theorem 4.1:** Let M be a right R-Module. Then M is EC-PQ-P-injective if and only if kers = keri,  $s, i \in S = End_R(M)$  implies Ss = Si.

**Proof:** Let  $s, i \in S$  with kers = keri. The map  $\phi: s(M) \to M$  define by  $\phi(s(m)) = i(m)$  for every  $m \in M$ . to show that  $\phi$  is essential monomorphism. Let  $s(m_1), s(m_2) \in s(M)$  such that  $\phi(s(m_1)) = \phi(s(m_2))$ . Then

$$\varphi s(m_1) = \varphi s(m_2) \Rightarrow i(m_1) = i(m_2)$$
 for every  $i \in M$ .

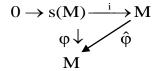
$$\Rightarrow$$
 i(m<sub>1</sub>) - i(m<sub>2</sub>) = 0  $\Rightarrow$  (m<sub>1</sub> - m<sub>2</sub>)  $\in$  ker  $i \Rightarrow$  m<sub>1</sub> - m<sub>2</sub>  $\in$  ker  $i =$  ker s  $\Rightarrow$  s(m<sub>1</sub> - m<sub>2</sub>) = 0

$$\Rightarrow$$
 s(m<sub>1</sub>) = s(m<sub>2</sub>)  $\Rightarrow$   $\varphi$ (s(m<sub>1</sub>)) =  $\varphi$ (s(m<sub>2</sub>))

$$\Rightarrow i(m_1) = i(m_2)$$

 $\Rightarrow \Phi$  is essential monomorphism.

Since m is EC-PQ-P-injective and s(M) is EC-M-cyclic submodule of M, there exists an R-homomorphism  $\hat{\phi}$ : $s(M) \rightarrow M$  such that  $\phi = \hat{\phi}$ .i



Where  $i:s(M)\to M$  is an inclusion monomorphism. Thus  $i=\phi.s=\hat{\phi}.i.s=\hat{\phi}.s\in Ss$ .

Then  $Si \subset Ss$  similarly  $Ss \subset Si$ . Therefore Ss = Si.

Conversely, obvious by lemma 1.1.

**Theorem 4.2:** Let M be EC-PQ-P-injective module. If A is a direct summand of M, then A is EC-PM-P-injective.

**Proof:** Let A be a direct summand of M. Let  $j: A \to M$  be injection mapping i.e.  $0 \to s(M) \xrightarrow{i} A \xrightarrow{j} M$  To show that  $\ker(j.i) = 0$ . Let  $s(m) \in \ker(j.i)$  for every  $m \in M$ . Then  $(j.i)(s(m)) = 0 \Rightarrow j(i(s(m)) = i(s(m)) = 0 \Rightarrow i(s(m)) = 0 \Rightarrow s(m) \in \ker(j.i) \Rightarrow s(m) = 0$ , (because i is monic). Then  $j.i: s(M) \to M$  is an essential monomorphism [5. pro 5.2]. Since M is a EC-PQ-P-injective and s(M) is EC-M-cyclic submodule of M, there exists an homomorphism  $\hat{i}: M \to M$  such that  $i.\alpha = \hat{i}.t$ , where  $t: s(M) \to M$  is the inclusion monomorphism. Let  $\pi: M \to A$  be projection map. Then  $\pi.j.i = \pi.\hat{i}.t$ . Since  $\pi.i = I_A$  and  $j = \pi.\hat{i}.t$ . Therefore  $\pi.\hat{i}$  is extension of  $\alpha$ . This shows A is EC-PM-P-injective. //

#### REFERENCES

- 1. A.K. Chaturvedi, B.M. Pandeya, A.J. Gupta, Quasi pseudo principally injective modules, Algebra Colloq. 16(3) (2009) 397-402.
- 2. A.K. Chaturvedi, B.M. Pandeya, A.J. Gupta, Modules whose M-cyclics are summand, Int. J. Algebra 3921) (2010) 1045-1049.
- 3. A.K. Chaturvedi, QP-injective and QPP-injective Modules, Southeast Asian Bull Math. 38 (2014) 191-104.
- 4. C.C Yucel, A note on ECS-modules, Palestine J. Math. 3(1) (2014) 383-387.
- 5. F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New-York, 1992.
- 6. H. Kalita, H.K. Sakiya, Pseudo p- injective modules and k-non singularity, Int. J. Math. Archiv 4(9) (2013) 233-236.
- 7. M.F. Yousif, W.K. Nicholsion, Principally Injective rings, J Algebra, 174 (1995), 77-93.
- 8. S. Baupradist, H.D. Hai, N.V. Sanh, on pseudo p-injectivity, Southeast Asian Bull Math. 35 (2011) (1) 21-27.
- 9. V. Camillo, Commutative rings whose principal ideals are annihilators, Portugal Math. 46. (1989) 33-37.
- 10. W.K. Nicholson, J.K. Park, M.F. Yousif, Principally quasi injective modules, Comm. Algebra 27(4) (1999) 1683-1693.
- **11.** Z. Zhu, Pseudo QP-injective modules and generalized pseudo QP-injective module, Int. Electron. J. Algebra 14(2013) 32-43.

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