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ON ECM-P-INJECTIVE MODULES

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ABSTRACT

The aim this article is to explore the characterization of M-cyclic submodule. Let R be a ring. M and N are R-modules. A module N is called ECM-principally injective module (briefly, ECM-P-injective) if every R-homomorphism from essentially M-cyclic submodule of M to N, can be extended to M. In this paper we obtain to investigate some characteristics of M-principally injective module. Using the notion EC-M-cyclic submodule of M.

Key words: *EC-M-cyclic, M-principally injective, ECM-principally injective, pseudo M-principally injective and pseudo quasi-principally injective.*

1. INTRODUCTION

Through this paper, by a ring R we always mean as associative with identity and every R-module is unitary. The notion principally injective module was introduced by Camollo [9]. Nicholson, park and Yousif studied the structure of principally injective and Quasi- principally injective modules [10]. N is called M- principally injective if Tansee and Wongwai also extended this notion. A right R-module every R-homomorphism from an M-cyclic submodule of M to N, can be extended to M. A module M is called Quasi- principally injective if it is M- principally injective. A submodule K of M is called essential submodule if $K \cap L \neq 0$ for every nonzero submodule L of M. In other words $K \cap N = 0 \Rightarrow K = 0$ (briefly; $K \leq^e M$). In this case M is called essential extension of K. A monomorphism $f : K \to M$ is said to be essential if $\mathrm{im} f \leq^e M$. A submodule K is called M-cyclic if K is image of element of S. ($S = \mathrm{End}_R(M)$ denotes endomorphism ring of M). A submodule K is called essentially M-cyclic (briefly; EC-M-cyclic) if it is the image of element of S and it's inclusion map is essential.

2. PRELIMINARY RESULTS

In this section, we study of essential submodule.

Lemma.2.1: Let M, N be right R-modules and let f: $N \to M$ be a homomorphism, if M' is an essential sub module of N, then $f^{-1}(M')$ is essential sub module of N.

Proposition 2.1: Let K and N be sub modules of an R-module M. Then

(i) $K \leq^e M \Leftrightarrow K \leq^e N$ and $N \leq^e M$.
(ii) $K \leq^e M \Leftrightarrow K \cap Rm \neq 0 \quad \forall\, 0 \neq m \in M$.
(iii) Given $K \subset N$ if $N/K \leq^e M/K$ then $N \leq^e M$.
(iv) $K \cap N \leq^e M \Leftrightarrow K \leq^e M$ and $N \leq^e M$.
(v) If $K \leq^e M$ then $K \cap N \leq^e M$.
(vi) $K \leq^e M \Leftrightarrow$ for each $0 \neq m \in M\ \exists$ an $r \in R$ such that $0 \neq mr \in K$.
(vii) $K_1 \oplus K_2 \leq^e M_1 \oplus M_2 \Leftrightarrow K_1 \leq^e M_1$ and $K_2 \leq^e M_2$ for each $K_1 \leq M_1 \leq M$ and $K_2 \leq M_2 \leq M$.
(viii) If $M = \oplus_{i=1}^{n} M_i$ and $K_i \leq M_i$ for each $i \in I$, then $= \oplus_{i=1}^{n} N_i \leq^e M$.

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Proof:

i) Given that $K \le N$ and $K \cap L = 0$, $K \le M$. Clearly $L \le N \le M \Rightarrow L \le M$. Since $K \le^e M$, then $K \cap L = 0 \Rightarrow L = 0$ implies $K \le^e N$. Let $T \le M$ such that $N \cap T = 0$, $K \cap T \le N \cap T = 0$ $\Rightarrow N \cap T = 0$, since $K \le^e M$, then $T = 0 \Rightarrow N \le^e M$. Conversely, Let $K \cap S = 0$ for some $S \le M$ $\Rightarrow K \cap (S \cap N) = 0$. Since $K \le^e N$ and $S \cap N \le N$, $S \cap N = 0$ with $N \le^e M \Rightarrow S = 0$ so $K \le^e M$.

ii) We have $N \cap K \le N \le M$, Since $N \cap K \le^e M \Rightarrow N \le^e M$. Similarly $N \cap K \le N \le M$, Since $N \cap K \le^e M \Rightarrow K \le^e M$. Similarly $N \le^e M$, $K \le^e M$. To prove $N \cap K \le^e M$. Let $(N \cap K) \cap T = 0$ for some $T \le M \Rightarrow N \cap (K \cap T) = 0$, since $N \le^e M \Rightarrow K \cap T = 0$ and also $K \le^e M$, therefore $T = 0$, by (i) we get $K \cap N \le^e M$.

iii) and iv) [1].

v). Let $(K \cap N) \cap T = 0$ for some $T \le K \le M$, therefore $K \cap (N \cap T) = 0$, since $K \le^e M$ and $N \cap T \le 0 \Rightarrow N \cap T = 0$ and $N \cap T = T \Rightarrow T = 0$, Hence $K \cap N \le^e N$.

vii) and viii) [1]. //

3. ECM-P-INJECTIVE MODULE

Definition 3.1: An R-module N is called essential M-principally injective module (ECM-P-injective), if every R-homomorphism from EC-M-cyclic submodule K of M to N, can be extended to M, in general the following diagram is commutative,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "h"'] & M \arrow[dl, "\phi"] \\ & N & \end{tikzcd}$$

Fig.-1

i.e. $\Phi.i = h$. where $\varphi \in End_R(M)$ and $K = \varphi(M) \le^e M$.

Example 3.1:

(i) Z is essential sub module of the Z-module Q, is cyclic, but not Q-cyclic, for every non zero homomorphism $f: Q \to Q$ is an epimorphism.

(ii) Let $M = Z_1 \oplus Z_2 \oplus Z_3$ is a z-module, since $M/Z_3 = Z_2 \oplus Z_2$, then $Z_2 \oplus Z_2$ is EC-M-cyclic, but $Z_2 \oplus Z_2$ is not cyclic.

(M-cyclic submodule and cyclic module both are completely different concepts)

Lemma 3.1: Let M and N be R-modules. Then N is ECM-P-injective if and only if for each $s \in S = End_R(M)$.

$$Hom_R(M,N)_S = \{f : Hom_R(M,N) : f(kers) = 0\}$$

Proof: Assume that N is ECM-P-injective module. We want to show that

$$Hom_R(M,N)_S = \{f : Hom_R(M,N) : f(kers) = 0\}$$

It is clear that $$Hom_R(M,N)_S \subseteq \{f : Hom_R(M,N) : f(kers) = 0\}$$

Let $f \in Hom_R(M,N)$ such that $f(kers) = 0 \Rightarrow kers \subset kerf.$ Then there is an homomorphism $i : s(M) \to M$ such that $i.s = f$. Since N is ECM-P-injective module.

$$\begin{tikzcd} 0 \arrow[r] & s(M) \arrow[r, "i"] \arrow[d, "h"'] & M \arrow[dl, "\phi"] \\ & N & \end{tikzcd}$$

Fig.-2

There exists an R-homomorphism $\phi : M \to N$ such that $\phi.i = h$ where the inclusion map $i : s(M) \to M$ is essential monomorphism with s(M) is large M-cyclic submodule of M. Then $\varphi s \in Hom_R(M,N)$ and kers $\in ker\varphi s$ $\Rightarrow \varphi s(kers) = 0$. By assumption $\varphi s(M) = u[s(M)] = u[i(s(M))] \Rightarrow us(M)$ is also large M-cyclic submodule of M. This show that N is ECM-P-injective module.//

Theorem 3.1: Let M and N be R-modules. Then M is N-Principally projective module and every EC-M-cyclic submodule of N is ECM-P-injective if and only if Nis ECM-P-injective module and EC-M-cyclic submodule of M is ECM-P-injective.

Proof: Let M be N-Principally projective module and suppose that every EC-M-cyclic submodule of N is ECM-P-injective. Since n is trivially M-cyclic, so N is ECM-P-injective. Let $\varphi \in \mathrm{End}_R(M)$.

Let $\nu : M \to L$ be small epimorphism and let $h : \varphi(M) \to L$ be any homomorphism, where $\varphi(M)$ is EC-M-cyclic submodule of M.

Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \varphi(M) \arrow[r, "i"] \arrow[dl, "g"'] \arrow[d, "h"] & M \arrow[dll] \arrow[dl, "l"] \\ N \arrow[r, "\nu"'] & L \arrow[r] & 0 \end{tikzcd}$$

Fig.-3

Where $i : \varphi(M) \to M$ is an inclusion monomorphism, implies $\varphi(M) \leq^e M$. we have L is M-cyclic i.e. L is ECM-injective. There exists an epimorphism $l : M \to L$ such that $l.i = h$ and the sequence $0 \to \varphi(M) \xrightarrow{i} M \xrightarrow{l} L \to 0$ is exact. Since M is N-projective module this implies, so there exists an homomorphism $t : M \to N$ such that $\nu.t = l$ and the map $g : \varphi(M) \to N$ such that $g = t.i$.

Now $\nu.g = \nu.t.i = l.i = h$. This shows that every M-cyclic sub module of M is N-P projective.

Conversely, suppose that every M-cyclic sub module of M is N-P projective and N is ECM-P-injective.

Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \varphi(M) \arrow[r, "i"] \arrow[dl, "g"'] \arrow[d, "h"] & M \arrow[dll] \arrow[dl, "l"] \\ N \arrow[r, "\nu"'] & B \arrow[r] & 0 \end{tikzcd}$$

Fig.-4

where $i : \varphi(M) \to M$ is inclusion monomorphism and $h : \varphi(M) \to B$ is any homomorphism, $g : M \to B$ is an required small epimorphism. Since $\varphi(M)$ is N-projective module, thus there exists a homomorphism $g : \varphi(M) \to N$ such that $\nu.g = h$. But N is ECM-P-injective, so there is an homomorphism $t : M \to N$ such that $t.i = g$, Define $l : M \to N$ by $l = \nu.t$. Now $l.i = \nu.t.i = \nu.g = h.$

Theorem 3.2: The following are equivalent for a projective module M.

(i) Every small M-cyclic sub module of M is projective.
(ii) Every factor module of an ECM-P-injective is ECM-P-injective.
(iii) Every factor module of an injective R-module is ECM-P-injective.

Proof:
(i) ⇒ (ii): Let N be an ECM-injective module, X is small M-cyclic sub module of N. let $s \in \mathrm{End}_R(M)$. Consider the diagram:

$$\begin{tikzcd} & s(M) \arrow[r, "i"] \arrow[d, "\hat{\varphi}"'] \arrow[dr, "\varphi"] & M \arrow[r] & 0 & \ker i \prec\prec s(M) \\ X \arrow[r] & N \arrow[r, "\eta"'] & \frac{N}{X} & & \end{tikzcd}$$

Fig.-5

Let $\varphi : s(M) \to N / X$ be an R-homomorphism by (i) there exists an R- homomorphism $\hat{\varphi} : s(M) \to N$ such that $\varphi = \eta.\hat{\varphi}$. Where $\eta : N \to N / X$ is the natural epimorphism. Since N is ECM-P-injective, there exists an R-homomorphism $t : M \to N$, which is essential extension of $\hat{\varphi}$ to M. Then $\mu.t$ is essential extension of to M i.e. factor module is ECM-P-injective.

(ii) ⇒(iii): Clear.

(iii) ⇒ (i): Let $s(M)$ be an small M-cyclic sub module of M and $h : A \to B$ is an epimorphism and let $\alpha : s(M) \to B$ be an homomorphism imbed A in an injective module E. $B \cong A/\ker h$ is a submodule of LMP-injective module $E/\ker h$. Let a map $\alpha : s(M) \to E/\ker h$ by hypothesis we can extend $\hat{\alpha} : M \to E/\ker h$. Since M is projective, $\hat{\alpha}$ can be lifted to $g : M \to E$ such that $\eta.g = \hat{\alpha}$ where η is natural map. It is clear that $g(s(M)) \subset A$. Therefore we have lifted α, Implies every small M- cyclic sub module of is projective.//

4. EC- PSEUDO QUASI- PRINCIPALLY INJECTIVE MODULE (EC-PQ-P-INJECTIVE MODULE)

Definition 4.1: Let M be right R-module. A right R-module N is called essentially pseudo-M-principally injective module (EC-PM-P- injective) if every R- monomorphism from EC-M-cyclic submodule of M to N can be extended to an $\mathrm{End}_R(M)$. The module M is called essentially pseudo Quasi- principally injective module.

Lemma 4.1: Every EC-X-cyclic submodule of X is an EC-M-cyclic submodule of M for every EC-M-cyclic submodule X of M.

Proof: [11].

Proposition 4.1: N is EC-PM-P-injective if and only if N is EC-PX-P-injective foe every EC-M-cyclic sub module of M.

Proof: ⇒ Let $X = s(M)$ is an EC-M-cyclic sub module of M. $t(X)$ is a EC-X-cyclic submodule of X and let $\varphi : t(X) \to N$ be an R-essential monomorphism. Since $t, s \in S$ and $t(M) = t(X)$. Since N is EC-PM-P-injective, there exists an R-homomorphism $\hat{\alpha} : M \to N$ such that $\alpha = \hat{\alpha}.t_1.t_2$

$$\begin{tikzcd}
t(X) \arrow[r, "t_2"] \arrow[dr, "\alpha"'] & X \arrow[r, "t_1"] \arrow[d] & M \arrow[dl, "\hat{\alpha}"] \\
& N &
\end{tikzcd}$$

Where $t_1 : X \to M$, $t_2 : t(X) \to X$ both are inclusion monomorphisms. Then $\hat{\alpha}.t_2$ is the extension of α. [5. pro. 5.12] N is EC-PM-P-injective.
⇐ it is clear.//

Theorem 4.1: Let M be a right R-Module. Then M is EC-PQ-P-injective if and only if $\ker s = \ker i$, $s, i \in S = \mathrm{End}_R(M)$ implies $Ss = Si$.

Proof: Let $s, i \in S$ with $\ker s = \ker i$. The map $\varphi : s(M) \to M$ define by $\varphi(s(m)) = i(m)$ for every m ∈M. to show that φ is essential monomorphism. Let $s(m_1), s(m_2) \in s(M)$ such that $\varphi(s(m_1)) = \varphi(s(m_2))$. Then

$\varphi s(m_1) = \varphi s(m_2) \Rightarrow i(m_1) = i(m_2)$ for every i ∈M.

$\Rightarrow i(m_1) - i(m_2) = 0 \;\Rightarrow\; (m_1 - m_2) \in \ker i \;\Rightarrow\; m_1 - m_2 \in \ker i = \ker s \;\Rightarrow\; s(m_1 - m_2) = 0$

$\Rightarrow s(m_1) = s(m_2) \;\Rightarrow\; \varphi(s(m_1)) = \varphi(s(m_2))$

$\Rightarrow i(m_1) = i(m_2)$

⇒ φ is essential monomorphism.

Since m is EC-PQ-P-injective and $s(M)$ is EC-M-cyclic submodule of M, there exists an R-homomorphism $\hat{\varphi} : s(M) \to M$ such that $\varphi = \hat{\varphi}.i$

$$\begin{tikzcd}
0 \arrow[r] & s(M) \arrow[r, "i"] \arrow[d, "\varphi"'] & M \arrow[dl, "\hat{\varphi}"] \\
& M &
\end{tikzcd}$$

Where $i : s(M) \to M$ is an inclusion monomorphism. Thus $i = \varphi.s = \hat{\varphi}.i.s = \hat{\varphi}.s \in Ss$.

Then $Si \subset Ss$ similarly $Ss \subset Si$. Therefore $Ss = Si$.

Conversely, obvious by lemma 1.1.

Theorem 4.2: Let M be EC-PQ-P-injective module. If A is a direct summand of M, then A is EC-PM-P-injective.

Proof: Let A be a direct summand of M. Let $j : A \to M$ be injection mapping i.e. $0 \to s(M) \xrightarrow{i} A \xrightarrow{j} M$ To show that $\ker(j.i) = 0$. Let $s(m) \in \ker(j.i)$ for every m ∈ M. Then $(j.i)(s(m)) = 0 \Rightarrow j(i(s(m)) = i(s(m) = 0 \Rightarrow i(s(m) = 0 \Rightarrow s(m) \in \ker i \Rightarrow s(m) = 0$, (because i is monic). Then $j.i : s(M) \to M$ is an essential monomorphism [5. pro 5.2]. Since M is a EC-PQ-P-injective and s(M) is EC- M-cyclic submodule of M, there exists an homomorphism $\hat{i} : M \to M$ such that $i.\alpha = \hat{i}.t$, where $t : s(M) \to M$ is the inclusion monomorphism. Let $\pi : M \to A$ be projection map. Then $\pi.j.i = \pi.\hat{i}.t$. Since $\pi.i = I_A$ and $j = \pi.\hat{i}.t$. Therefore $\pi.\hat{i}$ is extension of α. This shows A is EC-PM-P-injective. //

REFERENCES

1. A.K. Chaturvedi, B.M. Pandeya, A.J. Gupta, Quasi pseudo principally injective modules, Algebra Colloq. 16(3) (2009) 397-402.
2. A.K. Chaturvedi, B.M. Pandeya, A.J. Gupta, Modules whose M-cyclics are summand, Int. J. Algebra 3921) (2010) 1045-1049.
3. A.K. Chaturvedi, QP-injective and QPP-injective Modules, Southeast Asian Bull Math. 38 (2014) 191-104.
4. C.C Yucel, A note on ECS-modules, Palestine J. Math. 3(1) (2014) 383-387.
5. F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New- York, 1992.
6. H. Kalita, H.K. Sakiya, Pseudo p- injective modules and k-non singularity, Int. J. Math. Archiv 4(9) (2013) 233-236.
7. M.F. Yousif, W.K. Nicholsion, Principally Injective rings, J Algebra, 174 (1995), 77-93.
8. S. Baupradist, H.D. Hai, N.V. Sanh, on pseudo p-injectivity, Southeast Asian Bull Math. 35 (2011) (1) 21-27.
9. V. Camillo, Commutative rings whose principal ideals are annihilators, Portugal Math. 46. (1989) 33-37.
10. W.K. Nicholson, J.K. Park, M.F. Yousif, Principally quasi injective modules, Comm. Algebra 27(4) (1999) 1683-1693.
11. Z. Zhu, Pseudo QP-injective modules and generalized pseudo QP-injective module, Int. Electron. J. Algebra 14(2013) 32-43.

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