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IDEMPOTENT PROPERTY OF SEMIRINGS

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ABSTRACT

In this paper, it was proved that, if S is a totally ordered Idempotent semiring and (S, •) is positively totally ordered (negatively totally ordered), then ax = a or x.

Keywords: *Absorbing, K-regular semiring, Singular semigroup.*

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1. INTRODUCTION:

The first formal definition of semiring was introduced in the year 1934 by Vandiver. However the developments of the theory in semirings have been taking place since 1950. Semirings flourish in the mathematical world around us. A semiring is basic structure in Mathematics. Certainly the first mathematical structure we know–the natural number set N is a semiring. Other semirings take place naturally in different areas of mathematics as graph theory, functional analysis, commutative, non-commutative ring theory and the mathematical modeling of quantum physics and parallel computation systems.

The word idempotent signifies the study of semirings in which the addition operation is idempotent a + a = a. The best known example for idempotent semiring is the max-plus semiring. Interest has been shown in such structures arose in late 1950s through the observation that certain problems of discrete optimization could be linearised over suitable idempotent semirings. Idempotent semiring is a fundamental structure that has widespread applications in Computer Science. Idempotent semiring is a ring with additive idempotent.

2. PRELIMINARIES:

Idempotent semiring 2.1: A semiring S is said to be Idempotent if (S, +) and (S, •) are idempotents.

i.e $a + a = a$ and $a^2 = a$ for all a in S.

Singular semigroup 2.2: A semigroup (S, •) is left (right) singular if ax = a (ax = x) for all a, x in S. A semigroup (S, +) is left (right) singular if a + x = a (a + x = x) for all a, x in S.

K–regular semiring 2.3: In an additive idempotent semiring S if for each a in S there exist an element x in S such that a + axa = axa, then S is a K–regular semiring.

3. PROPERTIES OF IDEMPOTENT SEMIRING

Proposition 3.1: Let S be an Idempotent semiring.

a. If ax (a + x + ax) = ax and (S, •) is right singular, then (S, +) is right singular semigroup.
b. If 1 is a multiplicative identity and (S, •) is left cancellative, then 1 is an absorbing element.

Proof: (a) let us consider ax (a + x + ax) = ax for all a, x in S $\rightarrow$ **(1)**

By hypothesis (S, •) is right singular then ax = x and xa = x.

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From equation (1). We obtain $x (a + x + x) = x$
$\Rightarrow xa + x^2 + x^2 = x \Rightarrow a + x + x = x$
$\Rightarrow a + x = x$ for all a, x in S

Hence (S, +) is right singular semigroup.

(b) By the definition we have $a + a = a$ and $a^2 = a$ for all a in S.
$\Rightarrow a + a^2 = a + a \Rightarrow a + a^2 = a$
$\Rightarrow a (1 + a) = a.1$

By using left cancellation law we get $1 + a = 1$

Again let us take $a^2 = a$ for all a in S
$\Rightarrow a^2 + a = a + a \Rightarrow a (a + 1) = a (1 + 1)$
$\Rightarrow a (a + 1) = a.1$

By applying left cancellation property we obtain $a + 1 = 1$.

Thus $a + 1 = 1 + a = 1$ for all a, 1 in S.

Hence 1 is an absorbing element.

Theorem 3.2: Suppose S is an idempotent semiring and (S, +) is completely regular semigroup. Then (S, +) is singular semigroup.

Proof: By the definition of idempotent semiring we have $a + a = a$ for all a in S.

$\Rightarrow a + a + x = a + x \rightarrow$ **(1)**

Given that (S, +) is completely regular semigroup then $a + x = x + a$ and $a + x + a = a$. Now equation (1) becomes as $a + x + a = a + x \Rightarrow a + x = a$.

Thus (S, +) is left singular semigroup.

Since $a + x = x + a$

$\Rightarrow x + a = a$ for all a, x in S

Thus (S, +) is right singular semigroup.

Hence (S, +) is singular semigroup.

Proposition 3.3: If S is an idempotent semiring and (S, •) is left singular semigroup, then S is a K–regular semiring.

Proof: By hypothesis (S, •) is left singular, $ax = a$ for all a, x in S $\rightarrow$ **(1)**

$\Rightarrow a + ax = a + a \Rightarrow a + ax = a$

Using equation (1) in above we obtain $a + a (xa) = ax \Rightarrow a + axa = a (xa)$

Therefore $a + axa = axa$

Hence S is a K–regular semiring.

Theorem 3.4: Assume that S is an Idempotent semiring. Then (S, +) and (S, •) are weakly separative if and only if $a = x$.

Proof: Given that S is an idempotent Semiring, $a = a + a$ for all a in S.

Since (S, +) is weakly separative then $a + a = a + x = x + x$
$\Rightarrow a = a + x = x \Rightarrow a = x$

Suppose (S, •) is weakly separative then $a^2 = ax = x^2$

$\Rightarrow a = ax = x \ \Rightarrow a = x$

Conversely if $a = x$ then $a + a = a + x = x + x$.

Also $a = x$ implies $a^2 = ax = x^2$.

4. PROPERTIES OF TOTALLY ORDERED IDEMPOTENT SEMIRING

In this section we study the properties of totally ordered idempotent semiring.

Proposition 4.1: Suppose S is a totally ordered .idempotent semiring and (S, +) is positively totally ordered (negatively totally ordered). Then $a + x = a$ or x for a, x in S.

Proof: Assume that $a < x \Rightarrow a + x \leq x + x \ \rightarrow$ **(1)**

By the definition of idempotent semiring we have $a + a = a$ for all a in S.

Then equation (1) reduces to the form $a + x \leq x \rightarrow$ **(2)**

Since (S, +) p.t.o, $a + x \geq x \ \rightarrow$ **(3)**

Therefore from equations (2) and (3) we obtain $a + x = x$

If $x < a$ then $a + x \leq a + a$

$\Rightarrow a + x \leq a \ \rightarrow$ **(4)**

Again since (S, +) p.t.o, $a + x \geq a \rightarrow$ **(5)**

From equations (4) and (5) we conclude that $a + x = a$

Hence $a + x = a$ or x

Likewise we can prove that $a + x = x$ if (S, +) is negatively totally ordered.

Proposition 4.2: If S is a totally ordered Idempotent semiring and (S, •) is positively totally ordered (negatively totally ordered), then $ax = a$ or x.

Proof: Suppose $a < x \ \Rightarrow ax \leq x^2 \rightarrow$ **(1)**

By the definition of idempotent semiring $a^2 = a$ for all a in S.

Then equation (1) reduces to the form $ax \leq x \rightarrow$ **(2)**

Since (S, •) is p.t.o, $ax \geq x \ \rightarrow$ **(3)**

Therefore from equations (2) and (3) we obtain $ax = x$

Again let us consider $x < a$
$\Rightarrow ax \leq a.a \ \Rightarrow ax \leq a^2$
$\Rightarrow ax \leq a \ \rightarrow$ **(4)**

Since (S, •) is positively totally ordered, $ax \geq a \rightarrow$ **(5)**

From equations (4) and (5) we conclude that $ax = a$

Similarly we can show that $ax = a$ or x if (S, •) is negatively totally ordered.

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