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# LIE IDEALS AND LEFT JORDAN GENERALIZED DERIVATIONS OF PRIME RINGS

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**ABSTRACT**

*Let $R$ be a ring and $S$ a nonempty subset of $R$. An additive mapping $F: R \to R$ is called Left generalized derivation (Left Jordan derivation) on $S$. If there exists a derivation $d: R \to R$ such that $F(xy) = d(x)y + xF(y)$(respect to left Jordan generalized derivation $F(x^2) = d(x)x + xF(x)$) holds for all $x, y$ in $S$. Suppose that $R$ is a 2- torsion free prime ring and $U$ a non zero Lie ideal of $R$ such that $u^2$ in $U$ for all $u$ in. In this paper we proved that if $F$ is a left Jordan generalized derivation on $U$, then $F$ is a left generalized derivation on $U$*

***Key words:*** *Prime ring, Derivation, Generalized derivation, Left generalized derivation, Jordan generalizedderivation, Left Jordan generalized derivation, Lie ideals.*

## INTRODUCTION

Throughout this paper $R$will denote an associative ring with the Centre$Z(R)$. Recall that prime if $aRb = (0)$ implise that $a = 0$or$b = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. An additive subgroup $U$ of $R$ is said to be Lie ideal of $R$ if $[u, r] \in U$ for all $u \in U, r \in R$.We shall make use of Commutator identities: $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$. An additive mapping $d: R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$, $(d(x^2) = d(x)x + xd(x))$, holds for all $x, y \in R$.An additive mapping $F: R \to R$ is called a generalized derivation (resp.Jordan generalized derivation) if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ (resp.$F(x^2) = F(x)x + xd(x)$) for all $x, y \in R$. An additive mapping $F: R \to R$ is called a left generalized derivation (resp.left Jordan generalized derivation) if there exists a derivation $d: R \to R$ Suchthat $F(xy) = d(x)y + xF(y)$ (resp.$F(x^2) = d(x)x + xF(x)$) for all$x, y \in R$. Obviously, every derivation is a Jordan derivation. The converse is need not be true in general. Afamous result due to Herstein [11] states that every Jordan derivation on 2-torsion free prime ring is a derivation. A brief proof of this result is presented in [8]. Further, Awtar [5] generalized this result on Lie ideals. Hvala [12] states that every generalized derivation on a ring is a Jordan generalized derivation. But the converse statement does not holds in general. In [4] states that if$R$is a 2-torsion free prime ring and$U$ a non zero Lie ideal of $R$ such that every Jordan generalized derivation on $R$ is a generalized derivation. The aim of present paper $R$is a 2-torsion free prime ring and $U$ a non zero lie ideal of $R$ such that every left Jordan generalized derivation on $R$ is a left generalized derivation .

We begin with the following result which is essentially proved in [6].

**Lemma 1:** If $U \not\subseteq Z$ is a Lie ideal of a 2-torsion free prime ring $R$ and $a, b \in R$ such that $aUb = (0)$, then $a = 0$ $or$ $b = 0$.

We define a mapping $\delta: R^2 \to R$ such that $\delta(x, y) = F(xy) - d(x)y - xF(y)$

Now we see that $\delta(x, y + z) = \delta(x, y) + \delta(x, z)$and $\delta(x + y, z) = \delta(x, z) + \delta(y, z)$ for all $x, y, z \in R$.

By the definition

$$\begin{aligned}\delta(x, y + z) &= F\big(x(y + z)\big) - d(x)(y + z) - xF(y + z)\\ &= F(xy + xz) - d(x)y - d(x)z - xF(y) - xF(z)\\ &= F(xy) - d(x)y - xF(y) + F(xz) - d(x)z - xF(z)\\ &= \delta(x, y) + \delta(x, z)\end{aligned}$$

$\delta(x, y + z) = \delta(x, y) + \delta(x, z)$ for all $x, y, z \in R$.

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Similarly we have prove that $\delta(x+y,z)=\delta(x,z)+\delta(y,z)$ for all $x,y,z\in R$.

More over if $\delta$ is zero then $F$ is left generalized derivation on $R$.

**Lemma 2:** Let $R$ be a 2- torsion free ring and $U$ be a non zero Lie ideal of $R$ such that $u^2\in U$, for all $u\in U$. If $F: R\to R$ is an additive mapping satisfying $F(u^2)=d(u)u+uF(u)$ for all $u\in U$ then

1. $F(uv+vu)=d(u)v+uF(v)+d(v)u+vF(u)$, for all $u,v\in U$.
2. $F(uvu)=d(u)vu+ud(v)u+uvF(u)$, for all $u,v\in U$.
3. $F(uvw+.wvu)=d(u)vw+ud(v)w+uvF(w)+d(w)vu+wd(v)u+wvF(u)$for all $u,v,w\in U$.

**Proof:**

$i)\ F(u+v)^2=F((u+v)(u+v))=d(u+v)(u+v)+(u+v)F(u+v)$
$=d(u)u+d(u)v+d(v)u+d(v)v+uF(u)+uF(v)+vF(u)+vF(v)$.

$F(u+v)^2=d(u)u+d(u)v+d(v)u+d(v)v+uF(u)+uF(u)+vF(u)+vF(v)$. (1)

On the other hand, we have
$F(u+v)^2=F((u+v)(u+v))$
$=F(u^2+uv+vu+v^2)=F(u^2)+F(uv+vu)+F(v^2)$
$=d(u)u+uF(u)+F(uv+vu)+d(v)v+vF(v)$.

$F(u+v)^2=d(u)u+uF(u)+F(uv+vu)+d(v)v+vF(v)$. (2)

From (1) & (2), we have
$d(u)u+d(u)v+d(v)u+d(v)v+uF(u)+uF(v)+vF(u)+vF(v)$
$=d(u)u+uF(u)+F(uv+vu)+d(v)v+vF(v)$

$F(uv+vu)=d(u)v+uF(v)+d(v)u+vF(u)$.

ii) Let $W=F(u(uv+vu)+(uv+vu)u)$.

On one hand, we have
$W=d(u)(uv+vu)+uF(uv+vu)+d(uv+vu)u+(uv+vu)F(u)$
$W=d(u)uv+d(u)vu+a\big(d(a)b+aF(b)+d(b)a+bF(a)\big)+\big(d(a)b+ad(b)+d(b)a+bd(a)\big)a+abF(a)$
$+baF(a)$
$W=d(u)uv+d(u)vu+ud(u)v+u^2F(v)+ud(v)u+uvF(u))+(d(u)v+ud(v)+d(v)u+vd(v))u$
$+uvF(u)+vuF(u)$. (3)

On the other hand, we have
$W=F(u^2v+2uvu+vu^2)$.
$W=d(u^2)v+u^2F(v)+2F(uvu)+d(v)u^2+vF(u^2)$
$W=(d(u\ )u+ud(u))v+u^2F(v)+2F(uvu)+d(v)u^2+v(d(u)u+uF(u))$
$W=\big(d(u\ )u+ud(u)\big)v+u^2F(v)+2F(uvu)+d(v)u^2+vd(u)u+vuF(u)$.(4)

From (3) & (4), we have
$d(u)uv+d(u)vu+ud(u)v+u^2F(v)+ud(v)u+uvF(u))+d(u)vu+ud(v)u+d(v)u^2+vd(u)u+uvF(u)$
$+vuF(u)=d(u\ )uv+ud(u)v+u^2F(v)+2F(uvu)+d(v)u^2+vd(u)u+vuF(u)$
$2F(uvu)=\ 2d(u)vu+2ud(u)v+2\,uvF(v)$

Since $R$ is 2-torsion free, we get
$F(uvu)=d(u)vu+ud(u)v+uvF(u)$.

$(iii)$ Linearizing (ii) by replacing $u$ by $u+w$
$F\big((u+w)v(u+w)\big)=d(u+w)v(u+w)+(u+w)d(v)(u+w)+(u+w)vF(u+w)$

From L.H.S
$F\big((u+w)v(u+w)\big)=F\big((uv+wv)(u+w)\big)$
$=F(uvu+uvw+wvu+wvw)$
$=F(uvu)+F(uvw)+F(wvu)+F(wvw)$
$=d(u)vu+ud(v)u+uvF(u)+F(uvw+wvu)+d(w)vw+wd(v)w+wvF(w)$

$F\big((u+w)v(u+w)\big) = F(u)vu + uF(v)u + uvF(v) + F(uvw + wvu) + d(w)vw + wd(v)w + wvF(w).$ (5)

From R.H.S

$d(u+w)v(u+w) + (u+w)d(v)(u+w) + (u+w)vF(u+w)$
$= (d(u)+d(w))(vu+vw) + (ud(w)+wd(v))(u+w) + (uv+wv)(F(u)+F(w))$
$= d(u)vu + d(u)vw + d(w)uv + d(w)vw + ud(v)u + ud(v)w + wd(v)u + wd(v)w + uvF(u)$
$+wvF(u) + uvF(w) + wvF(w).$ (6)

From (5) & (6), we get

$d(u)vu + ud(v)u + uvF(u) + F(uvw+wvu) + d(w)vw + wd(v)w + wvF(w)$
$= d(u)vu + d(u)vw + d(w)vu + d(w)vw + ud(v)u + ud(v)w + wd(v)u + wd(v)w + uvF(u)$
$+ wvF(u) + uvF(w) + wvF(w)$

$F(uvw+wvu) = d(u)vw + ud(v)w + uvF(w) + d(w)vu + wd(v)u + wvF(u).$

**Lemma 3:** Let $R$ be a 2-torsion free ring and $U$ be a nonzero Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$.If $F: R \to R$ is an additive mapping satisfying $F(u^2) = d(u)u + uF(u)$, for all $u \in U$ , then $\delta(u,v)w[u,v] = 0$, for all $u, v, w \in U$.

**Proof:** Let $W = F(uvwvu + vuwuv)$
$= d(uv)wvu + uvd(w)vu + uvwF(vu) + d(vu)wuv + vuwF(uv)$
$= d(u)vwvu + ud(v)wvu + uvd(w)vu + uvwF(vu) + d(v)uwuv + vd(u)wuv + vud(w)uv$
$+vuwF(uv).$ (7)

On the other hand

$w = F(u(vwv)u + v(uwu)v)$
$= d(u)vwvu + ud(vwv)u + uvwvF(u) + d(v)uwuv + vd(uwu)v + vuwuF(v)$
$= d(u)vwvu + ud(v)wvu + uvd(w)vu + uvwd(v)u + uvwvF(u) + d(v)uwuv + vd(u)wuv$
$+vud(w)uv + uvwd(u)v + vuwuF(v).$ (8)

From (7) & (8), we have

$d(u)vwvu + ud(v)wvu + uvd(w)vu + uvwF(vu) + d(v)uwuv + vd(u)wuv + vud(w)uv(uv) + uvwF(uv)$
$= d(u)vwvu + ud(v)wvu + uvd(w)vu + uvwd(v)u + uvwvF(u) + d(v)uwuv + vd(u)wuv$
$+ vud(w)uv + uvwd(u)v + vuwuF(v)$

$uvwF(uv) + vuwF(uv) = uvwd(v)u + vuwvF(u) + vuwd(u)v + vuwuF(v)$

$uvw\big(F(vu) - d(v)u - vF(u)\big) + \big(F(uv) - d(u)v - uF(v)\big) = 0$

$uvw\delta(v,u) + vuw\delta(u,v) = 0$

We know that

$\delta(u,v) = -\delta(v,u)$

$(uv - vu)w\delta(u,v) = 0$

$[u,v]w\delta(u,v) = 0$ for all $u, v, w \in U$

**MAIN RESULT**

The main result of the present paper as follows

**Theorem 1:** Let $R$ be a 2-torsion free prime ring and $U$ a non zero Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. If $F$ is an additive mapping of $R$into itself satisfying $F(u^2) = d(u)u + uF(u)$ for all $u \in U$, then $F(uv) = d(u)v + uF(v)$ for all $u \in U$.

**Proof:** If $U$ is commutative lie ideal of $R$ i.e $[u,v] = 0$ for all$u, v \in U$, then use the same argument as used in the proof of lemma 1. 3 of [11], $U \subset Z$.Now by lemma 2(iii) we have

$F(uvw + wvu) = d(u)vw + ud(v)w + uvF(w) + d(w)uv + wd(v)u + wvF(u).$ (9)

Since $u^2 \in U$,for all $u \in U$ we find that$uv + vu \in U$ for all $u, v \in U$. This yields that $2uv \in U$ for all $u, v \in U$. As the ideal $U$ is commutative, in view of lemma 2 (i) we have
$2F(uvw + wvu) = F\big((2uv)w + w2(uv)\big)$
$2F(uvw + wvu) = 2(d(uv)w + uvF(w) + d(w)uv + wF(uv)$

This shows that for all $u, v \in U$.
$F(uvw + wvu) = d(u)vw + ud(v)w + uvF(w) + d(w)uv + wF(uv).$ (10)

Using (9) & (10) and using the fact of $uv = vu$ we obtain
$d(u)vw + ud(v)w + uvF(w) + d(w)uv + wd(v)u + wvF(u)$
$= d(u)vw + ud(v)w + uvF(w) + d(w)uv + wF(uv)$
$w\big(F(vu) - d(v)u - vF(u)\big) = 0$
$w\delta(u, v) = 0$ for all $u, v, w \in U.$ (11)

Now, replacing$w$ by $[w, r]$ in (11) and using (11), we get$wr\delta(u, v) = 0$ for all $u, v, w \in U$ and $r \in R$ and hence $UR\,\delta(u, v) = 0$ for all $u, v \in U$. since $U \neq 0$ and $R$ is prime the above expression yields that $\delta(u, v) = 0$ for all $u, v \in U$. Hence, we get the required result.

Hence, onward we shall assume that $U$ is a non commutative Lie ideal of $R$ i.e.$U \not\subseteq Z(R)$. By lemma 3 we have $[u, v]U\,\delta(u, v) = 0$ for all $u, v \in U$. Thus in view of lemma 1, we find that for each pair $u, v \in U$ either $[u, v] = 0$ or $\delta(u, v) = 0$ for$u \in U$. Let $U_1 = \{u \in U | [u, v] = 0\}$ *and* $U_2 = \{u \in U | \delta(u, v) = 0\}$. Hence $U_1$ and $U_2$ are additive subgroups of $U$ whose union is $U$ By Brauer's trick, we have either $U = U_1$ or $U = U_2$. Again by using the same method we find that either $U = \{u \in U | U = U_1\}$ or $U = \{u \in U | U = U_2\}$. Since $U$ is non-commutative, we find that $\delta(u, v) = 0$, for all $u, v \in U$i.e. $F$ is left generalized derivation on $U$.

**Corollary:** let $R$ be a 2-torsion free prime ring and $F: R \to R$ be a left Jordan generalized derivation. Then $F$ is a left generalized derivation on $R$.

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