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THE PRODUCT SPAN OF SUM SPAN OF A SUBSET OF A COMPLETELY BOUNDED ARTEX SPACE OVER A BI-MONOID

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ABSTRACT

When Sum Combination is introduced, it was introduced only for a finite subset of an Artex Space A over a bi-monoid M. Now the sum span for any subset of a completely bounded Artex space over a bi-monoid is defined. When Product Combination is introduced, it was introduced only for a finite subset of an Artex Space over a bi-monoid. Now the Product span for any subset of a completely bounded Artex space over a bi-monoid is defined. The product span of sum span of a subset of a completely bounded artex space over a bi-monoid is defined. Propositions were found and proved.

Keywords: *Bi-monoids, Artex Spaces over bi-monoids, Completely Bounded Artex Spaces over bi-monoids, Sum Combination, Sum Span, Product Combination, Product Combination, Product Span of Sum Span.*

I. INTRODUCTION

The algebraic system Bi-semi-group is more general to the algebraic system ring or an associative ring. Artex Spaces over Bi-monoids were introduced. As a development of Artex Spaces over Bi-monoids, SubArtex spaces of Artex spaces over bi-monoids were introduced.. From the definition of a SubArtex space, it is clear that not every subset of an Artex space over a bi-monoid is a SubArtex space. Some propositions which qualify subsets to become SubArtex Spaces were found and proved. Completely Bounded Artex Spaces over bi-monoids were introduced. It contains the least and greatest elements namely 0 and 1. When Sum Combination was introduced, it was in troduced only for a finite subset of an Artex Space A over a bi-monoid M. Now the sum span for any subset of a completely bounded Artex space over a bi-monoid is defined. When Product Combination was introduced, it was introduced only for a finite subset of an Artex Space over a bi-monoid. Now the Product span for any subset of a completely bounded Artex space over a bi-monoid is defined. Now sum combination, sum span, product combination and product span together give a new SubArtex Space namely product span of sum span of a subset of a completely bounded Artex space over a bi-monoid. It will be useful for the development of the theory of Artex Spaces over bi-monoids

II. PRELIMINARIES

2.1. Semi-group: A non-empty set S together with a binary operation. is called a Semi-group if for all a, b, c ϵ S, a.(b . c) = (a.b).c

2.2. Monoid: A non-empty set N together with a binary operation **.** is called a monoid if
(i) (i) for all a, b, c ϵ N, a.(b . c) = (a.b).c and
(ii) there exists an element denoted by e in N such that a.e = a = e.a, for all a ϵ N.

The element e is called the identity element of the monoid N.

2.3. Relation: Let S be a non-empty set. Any subset of S×S is called a relation in S.

If R is a relation in S, then R is a subset of S×S.

If (a,b) belongs to the relation R, then we can express this by aRb or by a ≤ b.

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Note: A relation may be denoted by $\leq$

2.4. Partial Ordering: A relation $\leq$ on a set P is called a partial order relation or a partial ordering in P if
(i) $a \leq a$, for all aϵP ie $\leq$ is reflexive,
(ii) $a \leq b$ and $b \leq a$ implies $a = b$ ie $\leq$ is anti-symmetric, and
(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$ ie $\leq$ is transitive.

2.5. Partially Ordered Set (POSET): If $\leq$ is a partial ordering in P, then the ordered pair (P, $\leq$) is called a Partially Ordered Set or simply a POSET.

2.6. Lattice: A lattice is a partially ordered set (L, $\leq$) in which every pair of elements a, b ϵ L has a greatest lower bound and a least upper bound.

The greatest lower bound of a and b is denoted by aΛb and the least upper bound of a and b is denoted by aV b

2.7. Lattice as an Algebraic System: A lattice is an algebraic system (L,Λ,V) with two binary operations Λ and V on L which are both commutative, associative and satisfy the absorption laws namely aΛ(aVb) = a and aV(aΛb) = a, for all a, b ϵ L

The operations Λ and V are called cap and cup respectively, or sometimes meet and join respectively.

2.8. Properties: We have the following properties in a lattice (L, Λ,V)

1. a Λ a = a	1'.a V a = a	(Idempotent Law)
2. a Λ b = b Λ a	2'.a V b = b V a	(Commutative Law)
3. (a Λ b) Λ c = a Λ (b Λ c)	3'.(a V b) V c = a V(b V c)	(Associative Law)
4. a Λ (a V b) = a	4'.a V (a Λ b) = a, for all a, b, c ϵ L	(Absorption Law)

2.9. Complete Lattice: A lattice is called a complete lattice if each of its nonempty subsets has a least upper bound and a greatest lower bound.

Every finite lattice is a complete lattice and every complete lattice must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the lattice and are denoted by 0 and 1 respectively.

2.10. Bounded Lattice: A lattice which has both elements 0 and 1 is called a bounded lattice. A bounded lattice is denoted by (L, Λ, V, 0, 1)

The bounds 0 and 1 of a lattice (L, Λ, V) satisfy the following identities.
For any aϵL, a V 0 = a a Λ 1 = a a V 1 = 1 a Λ 0 = 0

2.10.1. Example: For any set S, the lattice (P(S), $\subseteq$) is a bounded lattice. Here for each A, B ϵ P(S), the least upper bound of A and B is A$\cup$B and the greatest lower bound of A and B is A$\cap$B. The bounds in this lattice are φ, the empty set and S, the universal set.

2.11. Complemented Lattice: Let (L, Λ, V, 0, 1) be a bounded lattice. An element a'ϵL is called a complement of an element aϵL if aΛ a' = 0, a V a' = 1. A bounded lattice (L, Λ, V, 0, 1) is said to be a complemented lattice if every element of L has at least one complement. A complemented lattice is denoted by (L,Λ, V,' 0, 1).

2.11.1. Example: For any set S, the lattice (P(S), $\subseteq$) is a Complemented lattice.

For each A, B ϵ P(S), the least upper bound of A and B is A$\cup$B and the greatest lower bound of A and B is A$\cap$B.

The bounds in this lattice are φ, the empty set and S, the universal set.

Here for any A ϵ P(S), the complement of A in P(S) is S-A

2.12. Doubly Closed Space: A non-empty set D together with two binary operations denoted by + and . is called a Doubly Closed Space if (i) a.(b+c) = a.b + a.c and (ii) (a+b).c = a.c + b.c, for all a, b, c ϵ D

A Doubly closed space is denoted by (D, +, .)

Note-1: The axioms (i) a.(b+c) = a.b + a.c and (ii) (a+b).c = a.c + b.c, for all a, b, c ϵ D are called the distributive properties of the Doubly Closed Space.

Note-2: The operations + and **.** need not be the usual addition and usual multiplication respectively.

2.12.1. Example: Let N be the set of all natural numbers.

Then (N, +, .), where + is the usual addition and . is the usual multiplication, is a Doubly closed space.

Similarly (Z, + , .), (Q, + , .), (R, + , .) and (C, + , .) are all Doubly closed spaces.

2.12.2. Example: (Z, +, –), where + is the usual addition and **–** is the usual subtraction, is not a Doubly closed space.

2.13. Bi-semi-group: A Doubly closed space (S, +, .) is called a Bi-semi-group if + and . are associative in D.

2.13.1. Example 2.2.1: (N, + , .), (Z, + , .), (Q, + , .), (R, + , .), and (C, + , .), where + is the usual addition and **.** is the usual multiplication, are all Bi-semi-groups.

2.14. Bi-monoid: A Bi-semi-group (M, +, .) is called a Bi-monoid if there exist elements denoted by 0 and 1 in S such that a+0=a=0+a, for all aϵM and a.1=a=1.a, for all aϵM .

The element 0 is called the identity element of M with respect to the binary operation + and the element 1 is called the identity element of M with respect to the binary operation**.**

2.14.1. Example: Let W = {0, 1, 2, 3,…}.Then (W, +, .), where + is the usual addition and . is the usual multiplication, is a Bi-monoid.

2.14.2. Example: Let Q'= Q^+ ∪{0}, where Q^+ is the set of all positive rational numbers. Then (Q', +, .) is a bi-monoid.

2.14.3. Example: R'= R^+ ∪{0}, where R^+ is the set of all positive real numbers. Then (R', +, .) is a bi-monoid.

2.14.4. Example: (Z, + , .), (Q, + , .), (R, + , .), and (C, + , .), where + is the usual addition and **.** is the usual multiplication, are all Bi-monoids.

2.15. Artex Space Over a Bi-monoid: Let (M, +, .) be a bi-monoid with the identity elements 0 and 1 with respect to + and **.** respectively. A non-empty set A together with two binary operations ^ and v is said to be an Artex Space Over the Bi-monoid (M, +, .) if

1. (A, Λ , V) is a lattice and
2. for each mϵM, m≠0, and aϵA, there exists an element ma ϵ A satisfying the following conditions:
   (i) m(a Λ b) = ma Λ mb
   (ii) m(a V b) = ma V mb
   (iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
   (iv) (mn)a = m(na), for all m, nϵM, m≠0, n≠0, and a, bϵA
   (v) 1.a = a, for all a ϵ A.

Here, ≤ is the partial order relation corresponding to the lattice (A, Λ, V) .The multiplication ma is called a **bi-monoid multiplication with an artex element** or simply bi-monoid multiplication in A.

2.15.1. Example: Let W = {0, 1, 2, 3,…}.

Then (W, +, .) is a bi-monoid , where + and **.** are the usual addition and multiplication respectively.

Let Z be the set of all integers

Then (Z, ≤) is a lattice in which Λ and V are defined by a Λ b = minimum of {a, b} and a V b = maximum of {a, b}, for all a, b ϵ Z.

Clearly for each mϵW, m≠0, and for each aϵZ, maϵZ.
Also,

(i) m(a Λ b) = ma Λ mb
(ii) m(a V b) = ma V mb
(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
(iv) (mn)a = m(na)
(v) 1.a = a , for all m,nϵW, m ≠ 0, n ≠ 0 and a,bϵZ

Therefore, Z is an Artex Space Over the Bi-monoid (W, +, .)

2.15.2. Example: As defined in Example 2.15.1, Q, the set of all rational numbers is an Artex space over W

2.15.3. Example: As defined in Example 2.15.1, R, the set of all real numbers is an Artex space over W.

2.15.4. Example: Let $Q' = Q^+ \cup \{0\}$, where Q^+ is the set of all positive rational numbers.

Then (Q', + , .) is a bi-monoid. Now as defined in Example 2.15.1, Q, the set of all rational numbers is an Artex space over Q'

2.15.5. Example: $R' = R^+ \cup \{0\}$, where R^+ is the set of all positive real numbers. Then (R', +, .) is a bi-monoid.

As defined in Example 2.15.1, R, the set of all real numbers is an Artex space over R'

2.16. Properties

2.16.1. Properties: We have the following properties in a lattice (L , Λ,V)

1. a Λ a = a	1'.a V a = a
2. a Λ b = b Λ a	2'.a V b = b V a
3. (a Λ b) Λ c = a Λ (b Λ c)	3'.(a V b) V c = a V(b V c)
4. a Λ (a V b) = a	4'.a V (a Λ b) = a, for all a, b, c ϵ L

Therefore, we have the following properties in an Artex Space A over a bi-monoid M.

(i) m(a Λ a) = ma	(i)'.m(a V a) = ma
(ii) (m(a Λ b) =m(b Λ a)	(ii)'.m(a V b) = m(b V a)
(iii) m((a Λ b)Λ c)=m(a Λ(b Λ c))	(iii)'.m((a V b) V c) = m(a V(b V c))
(iv) m(a Λ (a V b)) = ma	(iv)'.m(a V (a Λ b)) = ma,

for all mϵM, m ≠ 0 and a, b, c ϵ A

2.17. SubArtex Space: Let (A, Λ, V) be an Artex space over a bi-monoid. (M, +, .). Let S be a nonempty subset of A. Then S is said to be a SubArtex Space of A if (S, Λ, V) itself is an Artex Space over M.

2.17.1. Example: As defined in Example 2.15.1, Z is an Artex Space over W = {0, 1, 2, 3,…..} and W is a subset of Z. Also W itself is an Artex space over W under the operations defined in Z. Therefore, W is a SubArtex space of Z.

2.18. Complete Artex Space over a bi-monoid: An Artex space A over a bi- monoid M is said to be a Complete Artex Space over M if as a lattice, A is a complete lattice that is each nonempty subset of A has a least upper bound and a greatest lower bound.

2.18.1. Remark: Every Complete Artex space must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the Artex space and are denoted by 0 and 1 respectively.

2.19. Lower Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0.

2.20. Upper Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1.

2.21. Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Bounded Artex Space over M if A is both a Lower bounded Artex Space over M and an Upper bounded Artex Space over M.

2.22. Completely Bounded Artex Space over a bi-monoid: A Bounded Artex Space A over a bi-monoid M is said to be a Completely Bounded Artex Space over M if (i) 0.a = 0, for all a ϵ A (ii) m.0 = 0, for all m ϵ M.

2.22.1. Note: While the least and the greatest elements of the Complemented Artex Space is denoted by 0 and 1, the identity elements of the bi-monoid (M, +, .) with respect to addition and multiplication are, if no confusion arises, also denoted by 0 and 1 respectively.

III. THE SUM SPAN OF A SUBSET OF AN ARTEX SPACE OVER A BI-MONOID

3.1. Sum Combination: Let (A, Λ, V) be an Artex Space over a bi-monoid (M, +, .). Let $a_1, a_2, a_3, \ldots, a_n \in A$. Then any element of the form $m_1a_1Vm_2a_2Vm_3a_3V \ldots\ldots V\ m_na_n$, where $m_i \in M$, is called a Sum Combination or Join Combination of $a_1, a_2, a_3, \ldots\ldots a_n$.

3.2. The Sum Span of a subset of an Artex Space over a Bi-monoid: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty finite subset of A. Then the Sum Span of W or Join Span of W denoted by S[W] is defined to be the set of all sum combinations of elements of W. That is, if
$W = \{a_1, a_2, a_3, \ldots\ldots a_n\}$, then $S[W] = \{m_1a_1Vm_2a_2Vm_3a_3V \ldots\ldots Vm_na_n / m_i \in M\}$.

3.3. PROPOSITIONS

Proposition 3.3.1: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty finite subset of A. Then W ⊆ S [W]

Proposition 3.3.2: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .). Let W and V be any two nonempty finite subsets of A. Then W ⊆ V implies P [W] ⊆ P [V].

Proposition 3.3.3: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .). Let W and V be any two nonempty finite subsets of A. Then P [W∪V] = P [W] V P [V] .

3.4. Examples

3.4.1. Example : Let $R' = R^+ \cup\{0\}$, where R^+ is the set of all positive real numbers and let W = {0,1,2,3,…..} (R', ≤) is a lattice in which Λ and V are defined by a Λ b = mini {a, b} and a V b = maxi {a, b}, for all a, b ϵ R'.

Here ma is the usual multiplication of a by m.

Clearly for each m ϵ W, m≠0, and for each a ϵ R', ma ϵ R'.

Also,
- (i) m(a Λ b) = ma Λ mb
- (ii) m(a V b) = ma V mb
- (iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
- (iv) (mn)a = m(na) , for all m, n ϵ W, m≠0, n≠0, and a, b ϵ R'
- (v) 1.a = a, for all a ϵ R'

Therefore, R' is an Artex Space Over the bi-monoid (W, +, .)

Generally, if Λ_1, Λ_2, and Λ_3 are the cap operations of A, B and C respectively and if V_1, V_2, and V_3 are the cup operations of A, B and C respectively, then the cap of A×B×C denoted by Λ and the cup of A×B×C denoted by V are defined
$x \Lambda y = (a_1,b_1,c_1) \Lambda (a_2,b_2, c_2) = (a_1 \Lambda_1 a_2 \Lambda_1 a_3, b_1 \Lambda_2 b_2 \Lambda_2 b_3, c_1 \Lambda_3 c_2 \Lambda_3 c_3)$ and
$x V y = (a_1,b_1,c_1) V (a_2,b_2, c_2) = (a_1 V_1 a_2 V_1 a_3, b_1 V_2 b_2 V_2 b_3, c_1 V_3 c_2 V_3 c_3)$

Here, Λ_1, Λ_2, and Λ_3 denote the same meaning minimum of two elements in R' and V_1, V_2, and V_3 denote the same meaning maximum of two elements in R'.

Therefore, $R'^3 = R' \times R' \times R'$ is an Artex Space over W, where cap and cup operations are denoted by Λ and V respectively.

Let S = { (1,0,0) } and let T = { (0,1,0) }

Now P [S] = {(m, 0, 0) / mϵR'} and P [T] = {(0, n, 0) / nϵR'}
P [S] V P [T] = {(m, 0, 0) / mϵR'} V {(0, n, 0) / nϵR}
= $\{(m\ V_1 0 , 0V_2 n, 0V_3 0)\}$
= {(m,n,0)} (since $mV_1 0$ = max.{m,0} = m, $0V_2 n$=max.{0,n}=n and $0V_3 0$=max.{0,0}=0)

P [S] V P [T] = {(m, n, 0) / m, nϵR'} (i)

Now S ∪ T = {(1, 0, 0), (0, 1,0)}

Let m, n ϵ M, m≠0, n≠0

Then m(1,0,0) V n(0,1,0) = (m,0,0) V (0,n,0)
= (m $V_1$0, 0 V_2 n, V_3 0)
= (m, n, 0) (since m$V_1$0 = max.{m,0} = m, 0V_2n=max.{0,n}=n and 0$V_3$0=max.{0,0}=0)

Therefore, P [S∪T] = {(m, n, 0) / m, nϵR'} (ii)

From equations (i) and (ii) we have P [S∪T] = P [S] V P [T]

3.4.2. Example: Let S = {(1, 0, 0)} and let T = {(1,0,0),(0,1,0)}

Then P [S] = {(a, 0, 0) / a ϵ R'} and P [T] = {(a, 0, 0), (0, b, 0) / a, b ϵ R'}

Therefore, P [S] ⊆ P [T].

3.5. THE PRODUCT OF SUBSETS OF AN ARTEX SPACE OVER A BI-MONOID: Let (A, Λ, V) be an Artex Space over a bi-monoid (M, +, .). Let S and T be subsets of the Artex Space A. Then the product of S and T denoted by S Λ T is defined by S Λ T = {s Λ t / s ϵ S and t ϵ T}

3.6. Product Combination: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M , + , .). Let a_1, a_2, a_3, ……. a_n ϵ A. Then any element of the form m_1a_1 Λ m_2a_2 Λ m_3a_3 Λ ……. Λ m_na_n , where m_i ϵ M, is called a Product Combination or Meet Combination of a_1, a_2, a_3, ……. a_n.

3.7. The Product Span of a Subset of a Completely Bounded Artex Space over a Bi-monoid: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty finite subset of A. Then the Product Span of W or Meet Span of W denoted by P[W] is defined to be the set of all product combinations of elements of W. That is, if W = {a_1, a_2, a_3, ……. a_n}, then P[W] = {m_1a_1 Λ m_2a_2 Λ m_3a_3 Λ ……. Λ m_na_n / m_i ϵ M}.

3.8. PROPOSITION

Proposition 3.8.1: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .). Let W and V be any two nonempty finite subsets of A. Then P[W∪V] = P[W] Λ P[V] .

3.9. Example: Let R' = R^+ ∪{0}, where R^+ is the set of all positive real numbers and let W = {0,1,2,3,…..}(R',≤) is a lattice in which Λ and V are defined by a Λ b = mini {a, b} and a V b = maxi {a, b}, for all a, b ϵ R'.

Here ma is the usual multiplication of a by m.

Clearly for each m ϵ W, m≠0, and for each a ϵ R', ma ϵ R'.

Also,
(i) m(a Λ b) = ma Λ mb
(ii) m(a V b) = ma V mb
(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a
(iv) (mn)a = m(na), for all m, n ϵ W, m≠0, n≠0, and a, b ϵ R'
(v) 1.a = a, for all a ϵ R'

Therefore, R' is an Artex Space Over the bi-monoid (W, +, .)

Generally, if Λ_1, Λ_2, and Λ_3 are the cap operations of A, B and C respectively and if V_1, V_2, and V_3 are the cup operations of A , B and C respectively, then the cap of A×B×C denoted byΛ and the cup of A×B×C denoted by V are defined
x Λ y =(a_1,b_1,c_1) Λ (a_2,b_2, c_2) = (a_1 $\Lambda_1$$a_2$ $\Lambda_1$$a_3$, b_1 $\Lambda_2$$b_2$ $\Lambda_2$$b_3$, $c_1$$\Lambda_3$$c_2$ $\Lambda_3$$c_3$) and
x V y =(a_1,b_1,c_1) V (a_2,b_2, c_2) = (a_1 $V_1$$a_2$ $V_1$$a_3$, b_1 $V_2$$b_2$$V_2$$b_3$, $c_1$$V_3$$c_2$$V_3$$c_3$)

Here, Λ_1, Λ_2, and Λ_3 denote the same meaning minimum of two elements in R' and V_1, V_2, and V_3 denote the same meaning maximum of two elements in R'

Therefore, R'^3 = R'×R'×R' is an Artex Space over W, where cap and cup operations are denoted by Λ and V respectively.

Let H = {(1, 0, 0)} and let T = {(0, 1, 0)}

Now P[H] = {(m,0,0) / mϵR'} and P[T] = {(0,n,0) / nϵR'}
P[H] Λ P[T] = {(m,0,0) / mϵR'} V {(0,n,0) / nϵR'}
= {(m $\Lambda_1$0, 0 Λ_2n, 0 $\Lambda_3$0)}
= {(0,0,0)} (since m $\Lambda_1$0 = mini.{m,0} = 0, 0 Λ_2n= mini.{0,n}=0 and 0 $\Lambda_3$0= mini.{0,0}=0)

P[H] Λ P[T] = {(0,0,0)} (i)

Now H ∪ T = {(1, 0, 0), (0, 1, 0)}

Let m, n ϵ M, m≠0, n≠0

Then m(1,0,0) Λ n(0,1,0) = (m,0,0) Λ (0,n,0)
= (m Λ_1 0, 0 Λ_2 n, Λ_3 0)
= (0, 0, 0) (since m $\Lambda_1$0 = mini.{m,0} =0, 0 Λ_2n= mini.{0,n}=0 and 0 $\Lambda_3$0= mini.{0,0}=0)

Therefore, P[H∪T] = {(0,0,0)} (ii)

From equations (i) and (ii) we have P[H∪T] = P[H] Λ P[T]

IV. THE PRODUCT SPAN OF SUM SPAN OF A SUBSET OF A COMPLETELY BOUNDED ARTEX SPACE OVER A BI-MONOID

4.1. The Product Span of Sum Span a Subset of a Completely Bounded Artex Space over a Bi-monoid: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty subset of A. Then the Sum Span of W or Join Span of W denoted by S[W] is defined to be S[W] = {m_1a_1Vm_2a_2Vm_3a_3V …… Vm_na_n / $m_i\epsilon$M and a_i ϵW}. The Product Span of W or Meet Span of W denoted by P[W] is defined to be P[W] = {m_1a_1 Λ m_2a_2 Λ m_3a_3 Λ ……. Λ m_na_n / m_i ϵ M and a_i ϵW}. Then P[S[W]] is Product Span of the Sum span S[W].

4.1.1 Note: Every element x of P[S[W]] is of the following form:
x = ($m_{11}a_{11}$V$m_{12}a_{12}$V …… V$m_{1p}a_{1p}$) Λ ($m_{21}a_{21}$V$m_{22}a_{22}$V …… V$m_{2k}a_{2k}$) Λ ………. Λ ($m_{r1}a_{r1}$V$m_{r2}a_{r2}$V….V$m_{nrq}a_{rq}$), where a_{ij} ϵ W and $m_{\mathbf{ij}}$ ϵ M.

4.2. PROPOSITIONS

Proposition: 4.2.1 Let (A,Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty subset of A. Then S[W] ⊆ P[S[W]].

Proof: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty subset of A.

Then S[W] = {m_1a_1Vm_2a_2Vm_3a_3V …… Vm_na_n / m_i ϵ M and a_i ϵW}.

Let **x** ϵ S[W]

Then x = m_1a_1Vm_2a_2Vm_3a_3V …… Vm_na_n , where $m_i\epsilon$M and a_i ϵW

Now every element of P[S[W]] is of the form
($m_{11}a_{11}$V$m_{12}a_{12}$V …… V$m_{1p}a_{1p}$) Λ ($m_{21}a_{21}$V$m_{22}a_{22}$V …… V$m_{2k}a_{2k}$) Λ ………. Λ ($m_{r1}a_{r1}$V$m_{r2}a_{r2}$V….V$m_{nrq}a_{rq}$), where a_{ij} ϵ W and $m_{\mathbf{ij}}$ ϵ M.

Take $m_{11} = m_1$, $m_{12} = m_2$, ….. , $m_{1p} = m_n$ if p = n
Take $m_{11} = m_1$, $m_{12} = m_2$, ….. , $m_{1n} = m_n$ and $m_{1n+1} = m_{1n+2} = m_{1n+3} = 0$ if p> n
Take $m_{11} = m_1$, $m_{12} = m_2$, ….. , $m_{1p} = m_p$ and $m_{1p+1} = m_{p+1}$ ….. $m_{1n} = m_n$ if p< n
and
Take $a_{11} = a_1$, $a_{12} = a_2$, ….. , $a_{1p} = a_n$ if p = n
Take $a_{11} = a_1$, $a_{12} = a_2$, ….. , $a_{1n} = a_n$ and if p> n
Take $a_{11} = a_1$, $a_{12} = a_2$, ….. , $a_{1p} = a_p$ and $a_{1p+1} = a_{p+1}$ ….. $a_{1n} = a_n$ if p< n

Also take $m_{ij} = 0$, for i ≥2

Then clearly x ϵ P[S[W]]
Hence, S[W] ⊆ P[S[W]].

Proposition: 4.2.2 Let (A,Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty subset of A. Then P[S[W]] is a SubArtex space of A.

Proof: Let (A, Λ, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .)

Let W be a nonempty subset of A.

The Sum Span of W denoted by S[W] is defined to be
S[W] = {$m_1a_1Vm_2a_2Vm_3a_3V \ldots Vm_na_n$ / $m_i \in M$ and $a_i \in W$}.

The Product Span of W denoted by P[W] is defined to be
P[W] = {$m_1a_1 \Lambda m_2a_2 \Lambda m_3a_3 \Lambda \ldots . \Lambda m_na_n$ / $m_i \in M$ and $a_i \in W$}.

Then P[S[W]] is Product Span of the Sum span S[W].

Claim: P[S[W]] is a SubArtex space of A.

Let x, y ϵ P[S[W]] and m, n ϵ M.

Now every element of P[S[W]] is of the form
$(m_{11}a_{11}Vm_{12}a_{12}V \ldots\ldots Vm_{1p}a_{1p}) \Lambda (m_{21}a_{21}Vm_{22}a_{22}V \ldots\ldots Vm_{2k}a_{2k}) \Lambda \ldots\ldots\ldots \Lambda (m_{r1}a_{r1}Vm_{r2}a_{r2}V\ldots.Vm_{nrq}a_{rq})$,
where $a_{ij} \in W$ and $m_{ij} \in M$.

Since (A,Λ, V) is a Completely Bounded Artex Space over the bi-monoid (M, +, .), A contains the least and the greatest elements namely 0 and 1.

Therefore, m_{ij} can necessarily be taken as 0.

Therefore, x and y are the combinations of products and sums of elements of W

Therefore, mx Λ ny is the combinations of products and sums of elements of W and mx V ny is the combinations of products and sums of elements of W.

Therefore, mx Λ ny ϵ P[S[W]] and mx V nyϵP[S[W]]

Hence, P[S[W]] is a Aub Artex Space of A.

V. CONCLUSION

Sum Combination, Sum Span, Product Combination, Product Span, Product Span of Sum Span of a subset of a Completely Bounded Artex space over a bi-monoid will motivate the researchers.

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