International Research Journal of Pure Algebra-5(11), 2015, 191-193
Available online through www.rjpa.info ISSN 2248–9037

KAPREKAR'S PROCESS ON FOUR-DIGIT NUMBERS COMPUTATIONAL APPROACH

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(Received On: 31-10-15; Revised & Accepted On: 23-11-15)

ABSTRACT

Let n be a four-digit number not having all digits the same. We apply Kaprekar's process to n and prove it terminates with the integer 6174 within the first seven iterations.

2010 Mathematics Subject Classification: *11y35.*

Key words and phrases: *Kaprekar, Four-digit numbers.*

1. INTRODUCTION

For a four-digit number n not having all digits the same, let $a_3a_2a_1a_0$ be the number obtained by writing the digits of n in descending order, and $a_0a_1a_2a_3$ the number obtained by writing the digits in ascending order.

Let $D_1 = a_3a_2a_1a_0 - a_0a_1a_2a_3$. For D_1 let $d_3d_2d_1d_0$ be the number obtained by writing the digits of D_1 in descending order, and $d_0d_1d_2d_3$ the number obtained by writing the digits in ascending order. Let
$D_2 = d_3d_2d_3d_0 - d_0d_1d_2d_3$.

When this process continue it called Kaprekar's process. We use computational approach to show the process terminates with the integer 6174 within the first seven iterations.

We explain the process through the following example. Let $n=4629$, then $a_3a_2a_1a_0 = 9642$ and $a_0a_1a_2a_3 = 2469$
$D_1 = 9642 - 2469 = 7173$ and $d_3d_2d_1d_0 = 7731$.
$D_2 = 7731 - 1377 = 6354$.
$D_3 = 6543 - 3456 = 3087$.
$D_4 = 8730 - 0378 = 8352$.
$D_5 = 8532 - 2358 = 6174$.

Thus the process terminated with the integer 6174 in the fifth iteration.

2. LEMMAS AND THEOREMS

Lemma 1: Let n be a four-digit number, if $a_3a_2a_1a_0$ is the number obtained by writing the digits of n in descending order, and $D_1 = a_3a_2a_1a_0 - a_0a_1a_2a_3$, then $D_1 = 999(a_3 - a_0) + 90(a_2 - a_1)$.

Proof: We have $a_3a_2a_1a_0 = 1000a_3 + 100a_2 + 10a_1 + a_0$, and $a_0a_1a_2a_3 = 1000a_0 + 100a_1 + 10a_2 + a_3$, thus $D_1 = a_3a_2a_1a_0 - a_0a_1a_2a_3 = 1000(a_3 - a_0) + 100(a_2 - a_1) + 10(a_1 - a_2) + (a_0 - a_3)$
$= 999(a_3 - a_0) + 90(a_2 - a_1)$.

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The importance of the above lemma lies in the fact that the number D_1 obtained in the first iteration doesn't depend on the numerical value of n but rather on the two differences $(a_3 - a_0)$ and $(a_2 - a_1)$.

Lemma 2: Let S be the set of all four-digit numbers not having all digits the same. For n, m belong to S let $a_3a_2a_1a_0$ and $b_3b_2b_1b_0$ be the two numbers obtained by writing the digits of n and m in descending order respectively, define the relation $(\sim)$ on S by $n \sim m \Leftrightarrow (a_3 - a_0) = (b_3 - b_0)$ and $(a_2 - a_1) = (b_2 - b_1)$, then $(\sim)$ is an equivalence relation.

The relation on the above lemma partitions the set S into mutually disjoint Equivalence classes. We will denote these classes by $[A, B]$, where the class $[A, B]$ represents all integers $n \in S$ satisfying $(a_3 - a_0) = A,\ (a_2 - a_1) = B.$

Hence if n represented by the class $[A, B]$, then $D_1 = 999(A) + 90(B).$

Theorem 1: The equivalence relation $(\sim)$ has exactly 54 equivalence Classes.

Proof: Let n **any** four-digit number not having all digits the same, let $a_3a_2a_1a_0$ be the number obtained by writing the digits of n in descending order. Since $a_0 \le a_1 \le a_2 \le a_3$, then $0 \le a_2 - a_1 \le a_3 - a_0$, that is in any class $[A, B]$ we have $0 \le B \le A.$ On the other hand since $a_0 < a_3$ we have $1 \le a_3 - a_0 \le 9.$ Thus for any class $[A, B]$ we have $1 \le A \le 9$ and $0 \le B \le A.$ Therefore number of the equivalence classes N is given by

$$N = \sum_{A=1}^{9} \sum_{B=0}^{A} 1 = \sum_{A=1}^{9} A + 1 = 2 + 3 + 4 + \ldots + 10 = 54.$$

Here is a list of all equivalence classes

[1,0], [1,1]
[2,0], [2,1], [2,2]
[3,0], [3,1], [3,2], [3,3]
[4,0], [4,1], [4,2], [4,3], [4,4]
[5,0], [5,1], [5,2], [5,3], [5,4], [5,5]
[6,0], [6,1], [6,2], [6,3], [6,4], [6,5], [6,6]
[7,0], [7,1], [7,2], [7,3], [7,4], [7,5], [7,6], [7,7]
[8,0], [8,1], [8,2], [8,3], [8,4], [8,5], [8,6], [8,7], [8,8]
[9,0], [9,1], [9,2], [9,3], [9,4], [9,5], [9,6], [9,7], [9,8], [9,9].

Upon the investigation of these classes we found out it is classified into seven groups, every group consists of classes need same number of iterations to terminate with the integer 6174.

Here a list of the seven groups.

$G_1 = [6,2]$
$G_2 = [4,2], [8,4], [8,6]$
$G_3 = [2,1], [3,1], [4,3], [6,3], [7,1], [7,4], [7,6], [8,1], [9,2], [9,3], [9,7], [9,8]$
$G_4 = [1,1], [2,0], [4,0], [4,4], [6,4], [6,6], [7,0], [9,0], [9,1], [9,9]$
$G_5 = [1,0], [3,2], [5,3], [5,4], [5,5], [6,5], [7,2], [7,5], [8,3], [8,7]$
$G_6 = [2,2], [3,0], [3,3], [5,0], [6,0], [7,3], [7,7], [8,0], [8,2], [8,8]$
$G_7 = [4,1], [5,1], [5,2], [6,1], [8,5], [9,4], [9,5], [9,6]$

Theorem 2: If n a four-digit number represented by a class from **the** groups G_1, G_2, G_3, then n terminates with the number 6174 within the first three iterations.

Proof: If n represented by the class [6,2] in G_1 then $D_1 = 999(6) + 90(2) = 6174$. Thus the number terminated in the first iteration.

If n represented by a class $[A, B]$ in G_2 then $D_1 = 999(A) + 90(B) = 4176,\ 8352,\ or\ 8532$. Thus D_1 represented by the class [6, 2] in G_1, therefore $D_2 = 999(6) + 90(2) = 6174$, hence n terminated in the second iteration.

If n represented by a class $[A, B]$ in G_3 then $D_1 = 999(A) + 90(B) = 2088,\ 3087,\ 4266,\ 7083,\ 7353,\ 7533,\ 8082,\ 9171,\ 9261,\ 9621,\ or\ 9711$. it follows that D_1 represented by one of the classes [8,6], [8,4] or [4,2] which all belong to G_2. Thus D_1 terminates in second iteration, and n terminates in 1+2=3 iterations.

Theorem 3: If n represented by a class in G_4, G_5, G_6 or G_7 then n terminates with the number 6174 within seven iterations.

Proof: If n represented by a class $[A, B]$ in G_4 then $D_1 = 999(A) + 90(B) = 1089,\ 1998,\ 3996,\ 4356,\ 6354,\ 6534,\ 6993,\ 8991,\ 9081,\ or\ 9801$. D_1 is represented by one of the following classes [9,7], [8,1], [6,3] or [3,1] and every these classes belong to G_3, then D_1 terminates in the third iteration, and n in the fourth.

If n represented by a class $[A, B]$ from G_5, then

$$D_1 = 999(A) + 90(B) = 0999,\ 3177,\ 5265,\ 5445,\ 6445,\ 5445,\ 6444,\ 7173,\ 7443,\ 8262,\ or\ 8622.$$

Then D_1 represented by one of the classes [9,0], [6,4], [4,0], [2,0] or [1,1] which all belong to G_4, then D_1 terminates in the fourth iterations, and n in the fifth one.

If n represented by a class $[A, B]$ from G_6 then

$$D_1 = 999(A) + 90(B) = 2178,\ 2997,\ 3267,\ 4994,\ 5994,\ 7263,\ 7623,\ 7992,\ 8172,\ or\ 8712.$$

Thus D_1 represented by one of the classes [7,5], [7,2], [5,3] or [5,4], and all of these classes belong to G_5. Hence D_1 terminates with 6174 in the fifth iteration and n terminates in the sixth.

Finally, if n represented by a class $[A, B]$ in G_7 then $D_1 = 999(A) + 90(B) = 4086,\ 5085,\ 5175,\ 6084,\ 8442,\ 9351,\ 9441,\ or\ 9531$. D_1 is represented by one of the classes [8,2], [8,0] or [6,0]. Since these classes belong to G_6 then D_1 terminates with the integer 6174 in the sixth iteration, and n in the seventh.

REFERENCES

1. A. Ayyad, An investigation of Kaprekar operation on six-digit numbers, computational approach, General Mathematics, Vol.20, No. 2-3(2012), 23-31.
2. D. R. Kaprekar, On kaprekar numbers, J. of Recr. Math. Vol.13 1980, 81-82.
3. K. E. Eldrige and S. Sagong, The determination of Kaprekar convergence and loop convergence of three-digit numbers, Amer. Math., Vol. 95, 1988, 105-112.

Source of Support: Nil, Conflict of interest: None Declared