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APPLICATIONS OF COPRIME POLYNOMIALS TO THE EQUATIONS OF RANKS OF MATRIXES OVER SKEW FIELD

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ABSTRACT

In this paper, we use the relatively prime polynomial of rank identities of the product of matrix, and the method of constructing block matrix, and conducting elementary transformation, if we can add some constraints and transform the rank identities from the domain to the skew field, finally we get a series of the conclusion of the rank of matrices over skew field.

Keywords: *skew field; coprime polynomials; equations of ranks.*

1. INTRODUCTION

Rank is an important and widely used numerical characteristic of matrix. Studying the ranks of matrixes is a basic problem of algebra. On the matrix theory over skew field, the result of the ranks of matrixes more takes the form of inequation. It causes a lot of inconvenience in the further application of the ranks of matrixes. Therefore, this article will discuss the identical equations of ranks of matrixes over skew field using the coprime polynomials.

2. PREPARATION KNOWLEDGE

In this paper, let K be a skew field and x be unknown. $K[x]$ is a ring. $r(A)$ is denoted the rank of matrix A. $d(x)=(f(x),g(x))$ and $m(x)=[f(x),g(x)]$ of leading coefficient being 1 are respectively denoted the greatest left common divisor and the least right common multiple $f(x)$ and $g(x)$. If $C(K)$ is the centre of skew field K, $C(K)=\{a\in K|ax=xa,\forall x\in K\}$.

Lemma 2.1: Let $f(x),g(x)\in C[x]$ be the polynomials of the leading coefficient being 1, and $d(x)$ and $m(x)$ are respectively denoted the greatest left common divisor and the least right common multiple of $f(x)$ and $g(x)$, then $f(x)g(x)=d(x)m(x)$.

Lemma 2.2: The sufficient and necessary condition for two polynomials $f(x),g(x)\in C[x]$ are said to be relatively prime is there are $u(x),v(x)\in K[x]$ such that $u(x)f(x)+v(x)g(x)=1$.

Lemma 2.3: Let $f_i(x),g_j(x)\in C[x],(i=1,2,\cdots,m;\ j=1,2,\cdots,n)$ and $(f_i(x),g_j(x))=1$, then

$$\left(f_i(x),\prod_{j=1}^{n}g_j(x)\right)=1,(i=1,2,\cdots,m;\ j=1,2,\cdots,n).$$

3. THEOREM AND COROLLARY

Theorem 3.1: Let $A\in K^{n\times n}$, $f(x),g(x)\in C[x]$. If $(f(x),g(x))=1$, then

$$r(f(A))+r(g(A))-r(f(A)g(A))=n.$$

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Proof: Since $(f(x), g(x)) = 1$, by the lemma 2.2, $u(x)f(x) + v(x)g(x) = 1, u(x), v(x) \in K[x]$. We substitute the matrix A into the formula, where $A \in K^{n\times n}$, then

$$u(A)f(A) + v(A)g(A) = E.$$

Constructing block matrix M and conducting elementary transformation, we have

$$M = \begin{pmatrix} f(A) & E \\ O & g(A) \end{pmatrix} \to \begin{pmatrix} f(A) & E - u(A)f(A) \\ O & g(A) \end{pmatrix}$$
$$\to \begin{pmatrix} f(A) & E - u(A)f(A) - g(A)v(A) \\ O & g(A) \end{pmatrix} = \begin{pmatrix} f(A) & O \\ O & g(A) \end{pmatrix}.$$

Then

$$r\begin{pmatrix} f(A) & E \\ O & g(A) \end{pmatrix} = r\begin{pmatrix} f(A) & O \\ O & g(A) \end{pmatrix} = r(f(A)) + r(g(A)). \tag{1}$$

We conduct another transformation for M,

$$M = \begin{pmatrix} f(A) & E \\ O & g(A) \end{pmatrix} \to \begin{pmatrix} f(A) & E \\ -f(A)g(A) & g(A) \end{pmatrix} \to \begin{pmatrix} O & E \\ -f(A)g(A) & O \end{pmatrix},$$

Therefore

$$r\begin{pmatrix} f(A) & E \\ O & g(A) \end{pmatrix} = r\begin{pmatrix} O & E \\ -f(A)g(A) & O \end{pmatrix} = n + r(f(A)) + r(g(A)). \tag{2}$$

By combining the above (1), (2) two equations, we have

$$r(f(A)) + r(g(A)) - r(f(A)g(A)) = n.$$

Specially, for the theorem 3.1, when $f(A)g(A) = O$, we have $r(f(A)) + r(g(A)) = n$.

Corollary 3.1: Let $A \in K^{n\times n}$, $f_i(x), g_j(x) \in K[x], (i = 1,2,\cdots,m; j = 1,2,\cdots,n)$. If $(f_i(x), g_j(x)) = 1$, then

$$r\left(\prod_{i=1}^{m} f_i(x)\right) + r\left(\prod_{j=1}^{n} g_j(x)\right) - r\left(\prod_{i=1}^{m} f_i(x) \prod_{j=1}^{n} g_j(x)\right) = n.$$

Proof: $(f_i(x), g_j(x)) = 1$, so $\left(f_i(x), \prod_{j=1}^{n} g_j(x)\right) = 1, (j = 1,2,\cdots,n)$, and then

$$\left(\prod_{i=1}^{m} f_i(x), \prod_{j=1}^{n} g_j(x)\right) = 1, (i = 1,2,\cdots,m).$$

According to the conclusion of theorem 3.1, we have

$$r\left(\prod_{i=1}^{m} f_i(x)\right) + r\left(\prod_{j=1}^{n} g_j(x)\right) - r\left(\prod_{i=1}^{m} f_i(x) \prod_{j=1}^{n} g_j(x)\right) = n.$$

Corollary 3.2: Let $A \in K^{n\times n}$, $f_i(x) \in C[x]$, $i = 1,2,\cdots,m$. If $f_1(x), f_2(x), \cdots, f_m(x)$ are said to be the polynomial of pairwise coprime, then

$$\sum_{i=1}^{m} r(f_i(A)) - r\left(\prod_{i=1}^{m} f_i(A)\right) = (m-1)n.$$

Proof: We use mathematical induction to prove the conclusion.
When $m = 2$, then $r(f_1(A)) + r(f_2(A)) = n + r(f_1(A)f_2(A))$ obviously established.

If when $m=k$, the conclusion is established. Then when $m=k$, set $g(x)=f_k(x)f_{k+1}(x)$, by the following two equations $(f_i(x),f_k(x))=1,(f_i(x),f_{k+1}(x))=1,(1\le i\le k-1)$, we have

$$(f_i(x),f_k(x)f_{k+1}(x))=(f_i(x),g(x))=1.$$

According to the hypothesis, we have

$$r(f_1(A))+r(f_2(A))+\cdots+r(f_{k-1}(A))+r(g(A))-r\left(\prod_{i=1}^{k+1}f_i(A)\right)=(k-1)n,$$

and know

$$r(f_k(A))+r(f_{k+1}(A))=n+r(f_k(A)f_{k+1}(A))=n+g(A).$$

So we will get the conclusion.

Specially, when $\prod_{i=1}^{m}f_i(x)=0$, we have $r(f_1(x))+r(f_2(x))+\cdots+r(f_m(A))=(m-1)n$.

Therefore we get the following corollary.

Corollary 3.3: Let $A\in K^{n\times n}$, $f_i(x)\in C[x]$ $i=1,2,\mathrm{L},m$. If $f_1(x),f_2(x),\cdots,f_m(x)$ are said to be the polynomial of pairwise coprime, and $f_i(A)f_j(A)=0,i\ne j,(i,j=1,2,\cdots,m)$, we have

$$r(f_1(A))+r(f_2(A))+\cdots+r(f_m(A))=\frac{1}{2}mn.$$

Proof: Because $f_i(A)f_j(A)=O,(f_i(A),f_j(A))=1,i\ne j$,

$$r(f_i(A))+r(f_j(A))=n,i\ne j,(i,j=1,2,\cdots,m).$$

We will add $m(m-1)$ polynomials above and get the conclusion:

$$2(m-1)(r(f_1(x))+r(f_2(x))+\cdots+r(f_m(A)))=m(m-1)n.$$

So the corollary 3.3 was established.

Theorem 3.2: Let $A\in K^{n\times n}$, $f(x),g(x)\in C[x]$, and be the above definition of $d(x),m(x)$, then

$$r(f(A))+r(g(A))=r(d(A))+r(m(A)).$$

Proof: If there are zero polynomials at least between $f(x)$ and $g(x)$, then the conclusion is obviously established.

Now let $f(x)\ne 0,g(x)\ne 0$, we have

$$f(x)=d(x)f_1(x),g(x)=d(x)g_1(x),(f_1(x),g_1(x)\in C[x]),(f_1(x),g_1(x))=1.$$

Then

$$d(A)=u(A)f(A)+v(A)g(A).$$

Constructing block matrix $\begin{pmatrix} f(A) & O \\ O & g(A)\end{pmatrix}$ and conducting elementary transformation, we have

$$\begin{pmatrix} f(A) & O \\ O & g(A)\end{pmatrix}\to\begin{pmatrix} f(A) & O \\ u(A)f(A)+v(A)g(A) & g(A)\end{pmatrix}$$

$$\to\begin{pmatrix} f(A) & O \\ d(A) & g(A)\end{pmatrix}\to\begin{pmatrix} f(A) & -g_1(A)f(A) \\ d(A) & g(A)-g_1(A)d(A)\end{pmatrix}$$

$$\to\begin{pmatrix} f(A)-d(A)f_1(A) & -g_1(A)f(A)+g_1(A)d(A)f_1(A)+g(A)f_1(A) \\ d(A) & g(A)-g_1(A)d(A) \end{pmatrix}$$

$$\to\begin{pmatrix} O & -g_1(A)d(A)f_1(A) \\ d(A) & O \end{pmatrix}\to\begin{pmatrix} d(A) & O \\ O & g_1(A)f_1(A)d(A) \end{pmatrix}=\begin{pmatrix} d(A) & O \\ O & m(A) \end{pmatrix}.$$

So,

$$r\begin{pmatrix} f(A) & O \\ O & g(A) \end{pmatrix}=r\begin{pmatrix} d(A) & O \\ O & m(A) \end{pmatrix}.$$

Therefore, the conclusion

$$r(f(A))+r(g(A))=r(d(A))+r(m(A))$$

was established.

Specially, if $f(A)=O$, then $r(f(A))=r(g(A))$.

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