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MATRIX FIELD OF FINITE AND INFINITE ORDER

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ABSTRACT

In the literature one can find several examples of a matrix group as well as matrix ring. However examples of a matrix field are generally not found. We provide some examples of matrix fields of finite as well as infinite order. In addition this article provides a technique to obtain finite matrix fields of order p for every positive prime p.

MSC2010: *12Exx, 12E20.*

Key-Words: *Algebra, binary operation, matrix group, matrix ring, matrix field, Galois field.*

INTRODUCTION

A non empty set R together with two binary operations '$+$' and '$\cdot$' is called a ring if

(1) $(R,+)$ is an Abelian group.

(2) $(R,.)$ is a semigroup.

(3) $a.(b+c)=a.b+a.c$ and $(b+c).a=b.a+c.a$, $\forall a,b,c \in R$.

If in addition

(4) $a.b=b.a, \forall a,b \in R$

A ring R having this property is known as a commutative ring.

(5) $\exists$ an element 1 in R such that

$1.a=a.1=a, \forall a \in R$

1 is called the multiplicative identity of R.

(6) for every non-zero element b in R $\exists$ an element c in R such that $b\cdot c=1$

then R is known as a field.

In the literature of abstract (modern) algebra ([1], [2], [3], [4], [5]) it is not very common to find examples of a field of matrices. The purpose of this article is to provide few examples of a field of matrices.

Condition (6) asserts that in a field R the product of any two non-zero elements is never zero. However in the case of a ring R one may find two non-zero elements b and c in R such that $b\cdot c=0$. A ring R having this property is known as a ring with zero divisors. The ring $M=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}: a,b,c,d \in K\right\}$ where K denotes the set of all rational (real or complex) numbers, is an example of a ring with zero divisors under the ordinary addition and multiplication of matrices.

It is well known that the ring of all square matrices of order 2 is a non- commutative ring with zero-divisors. Due to this it does not form a field and one does not generally think about matrix field. However we shall provide some examples of matrix fields in the next sections.

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This Ring M contains several matrix fields. All the examples of infinite matrix field given in the next section are subsets of the above ring M .

SOME EXAMPLES OF MATRIX FIELD OF INFINITE ORDER

Example 1: Let $F = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in K \right\}$. It is easy to see that F is a ring under usual addition and multiplication of matrices. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity of this ring. It is a commutative ring and $\forall A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in F$ with $a \neq 0$ we can find $B = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \in F$ such that $AB = I$. Therefore F is a field with respect to usual addition and multiplication of matrices.

Let Q , R and C denote the field of rational, real and complex numbers respectively.

Let $F_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in Q \right\}$,

$$F_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in R \right\}$$

and

$$F_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in C \right\}$$

then F_2 is an extension of F_1 and F_3 is an extension of F_2 .

It is known from ring theory that every ring R has a centre $Z(R)$. If R is an T -algebra where T is a field then $Z(R)$ contains a copy of the field T . One may conclude that field F given in this example is the centre of ring M given above. However it may be noted that the following two fields are not the centre of M but both are subsets of M .

Example 2: Let $F = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in K \right\}$. One can easily verify that F is a ring with respect to addition and multiplication of matrices. $I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the multiplicative identity of this ring. It is a commutative ring and $\forall A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \in F$ with $a \neq 0$ we can find $B = \begin{pmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{pmatrix} \in F$ such that $AB = I$. Therefore F is a field with respect to usual addition and multiplication of matrices.

Example 3: Let $F = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in K \right\}$. It is a ring with respect to addition and multiplication of matrices. $I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the multiplicative identity of this ring. It is a commutative ring and $\forall A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in F$ with

$a \neq 0,$ we can find $B = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix} \in F$ such that $AB = I$. Therefore F is a field with respect to usual addition and multiplication of matrices.

SOME EXAMPLES OF MATRIX FIELD OF FINITE ORDER

Let p be a prime number. Then $Z_p = \{0,1,2,3,4,5...p-1\}$ is a field under addition and multiplication modulo p. Using this field we can obtain different matrix fields of order p for every positive prime p. We shall consider only few different matrix fields of prime order. These fields provide matrix representations for Galois field of prime order. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ are any two 2×2 matrices defined over Z_p then the sum of A and B is defined as

$$A + B = (A + B) \bmod p = \begin{bmatrix} (a_1 + b_1) \bmod p & (a_2 + b_2) \bmod p \\ (a_3 + b_3) \bmod p & (a_4 + b_4) \bmod p \end{bmatrix}.$$

Similarly we can define the product of A and B. In the following examples we shall consider these operations for matrix addition and multiplication respectively.

Example 1: Let $F = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in Z_2 \right\}$. Then F is a finite field of order two with respect to addition and multiplication of matrices modulo 2. If we replace Z_2 by Z_3, then we will get a matrix field of order three. Similarly we can find a finite matrix field of higher order.

Example 2: If we take $F = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_2 \right\}$ then we will get a matrix field of order two. By replacing Z_2 with Z_3 we shall get a matrix field of order three. Similarly we can get a matrix field of order five, seven and eleven etc..

Example 3: By taking $F = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in Z_3 \right\}$ we can get a matrix field of order three. If we replace Z_3 by Z_5 then we will get a finite matrix field of order five. The identity element of this field will be given by $I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$.

Similarly if we replace Z_2 by Z_p in the above examples then we shall get finite matrix fields of order p. In the same way we can get a finite matrix field of order $p(\neq 2)$ from example 3. Thus this article provides a technique to obtain matrix representations for a finite field of prime order. One may find several such representations but all such fields are algebraically equivalent for a given prime p.

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