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A CHARACTERIZATION OF ALTERNATING GROUP A_{11} BY ITS CHARACTER DEGREE GRAPH

YONG YANG*

School of science, Sichuan University of Science and Engineering, Zigong Sichuan, 643000, P. R. China.

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ABSTRACT

In this paper, we give a new characterization of alternating group A_{11} by its character degree graph and order.

Key words: *Character degree graph, simple group, alternating group.*

MSC: *20C33, 20C15.*

1. INTRODUCTION

Let G be a finite group, Irr(G) be the set of irreducible characters of G, and cd(G) the set of degree of characters of G.

The most widely studied graph is the graph $\Gamma(G)$ whose vertices are the prime divisors of the character degrees of the group G and two vertices are joined by an edge if the product of the primes divides some character degree of G.

Recently more attention is paid to the graph of character degree of G and some new results are gotten. In [1], the authors proved that PSL$(2,p^2)$ is unique determined by the structure of its group algebra. Also in [2], simple groups whose orders are less than 6000 are considered by using the graph of character degree of group G.

As the development of this topics, we give a new characterization alternating group A_{11} by its character degree graph and order.

MAIN THEOREM

Let G be a group. If $\Gamma(G)=\Gamma(A_{11})$ and $|G|=\dfrac{11!}{2}$, then one of the following statements holds:

(1) G is isomorphic to a product HM_{22} of H by M_{22}.
(2) G is isomorphic to A_{11}.

2. SOME LEMMAS

In the following, we give some lemmas which will be used to prove the main result.

Lemma 1: Let $A \lhd G$ be abelian. Then $\chi(1) \mid\mid G:A|$ for all $\chi \in Irr(G)$.

Proof: See [3].

Lemma 2: Let G be a nonsolvable group. Then G has a normal series $1 \lhd H \lhd K \lhd G$ such that K/H is direct product of isomorphic nonabelian simple group and $|G/K| \mid |Out(K/H)|$.

****Corresponding Author: Yong Yang****
School of science, Sichuan University of Science and Engineering,
Zigong Sichuan, 643000, P. R. China.

Proof: See [4].

Lemma 3: Let G be a finite soluble group of order $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where $p_1, p_2, \cdots, p_n$ are primes. If $kp_n + 1 \nmid p_i^{a_i}$ for all $i \le n-1$ and $k > 0$, then the Sylow p_n-subgroup is normal in G.

Proof: See [5].

Lemma 4: If S is a finite non-abelian simple groups such that $11 \in \pi(S) \subseteq \{2,3,5,7,11\}$, then G is isomorphic to one of the simple groups listed in Table 1.

Proof: See [6].

Table-1: Finite non-abelian simple groups S with $11 \in \pi(S) \subseteq \{2,3,5,7,11\}$

S	|S|	Out(S)	S	|S|	Out(S)
$L_2(11)$	$2^2.3.5.11$	3	HS	$2^9.3^2.5^3.7^2.11$	2
M_{11}	$2^4.3^2.5.11$	1	$U_5(2)$	$2^{10}.3^5.5.11$	2
M_{12}	$2^6.3^3.5.11$	2	A_{12}	$2^9.3^5.5^2.7.11$	2
M_{22}	$2^7.3^2.5.7.11$	2	McL	$2^7.3^6.5^3.7.11$	2
A_{11}	$2^7.3^4.5^2.7.11$	2	$U_6(2)$	$2^{15}.3^6.5^2.7.11$	S_3

3. THE PROOF OF MAIN THEOREM

In the following, we give the proof of **Main Theorem**.

Proof: It is easy to get from [7] that cd(G)={1, 10, 44, 45, 110, 120, 126, 132, 165, 210, 231, 330, 385, 462, 550, 594, 660, 693, 825, 924, 990, 110, 1155, 1232, 1320, 1540, 2310}. It follows that the graph $\Gamma(G)$ of G is complete and has the vertex set {2, 3, 5, 7, 11}.

The results $O_7(G) = 1$ and $O_{11}(G) = 1$ will be shown. Assume $O_{11}(G) \ne 1$. In $\Gamma(G)$, there is an edge between the vertices 5 and 11. It follows that there is a character $\chi \in Irr(G)$ such that $5.11 \mid \chi(1) \mid\mid G : O_{11}(G) \mid$, contradicting Lemma 1. Second, assume that $O_7(G) \ne 1$. Since the graph $\Gamma(G)$ is complete, then the vertices 5 and 7 are connected. Thus there is an irreducible character χ such that $5.7 \mid \chi(1) \mid\mid G : O_7(G) \mid$, a contradiction. So $O_7(G) = 1$.

We will show that G is nosoluble group. Assume that G is soluble. Then there is an elementary minimal abelian p-group M. Since $O_7(G) = 1$ and $O_{11}(G) = 1$, then $p \in \{2,3,5\}$.

Let p=5. Then $|M| = 5$ since if $|M| = 5^2$, then there is no character χ such that $\chi(1) \mid\mid G : M \mid$, contradicting that the graph $\Gamma(G)$ of G is complete. Let H / M be a Hall $\{2,3,7,11\}$-subgroup of G / M 。 Then $|G / H| = 5$. It follows that $\frac{G}{H_G} \mapsto S_5$, where $H_G = \cap_{g \in G} H^g$ and so $7,11 \mid\mid H_G \mid$. By Lemma 3, we have that H_G is nilpotent and so G_7 is characteristic in H_G. Thus, G_7 is normal in G, a contradiction.

Let p=3. Then $|M| = 3^a$, where $a \in \{1,2,3\}$ as there is an edge between the vetices 3 and 11. Thus by [8], $\frac{N_G(M)}{C_G(M)}$ is isomorphic to a subgroup of GL(a, 3). It is easy to get from [7] that |GL(a,3)|=$3^{\frac{a(a-1)}{2}} (3^a - 1) \cdots (3^2 - 1)$. Therefore, the primes 2, 5, 7 and 11 are the prime divisors of the order of $C_G(M)$. If $N_G(M) = C_G(M)$, then G has a normal 3-complement H and $|H| = 2^7 \cdot 5^2 \cdot 7 \cdot 11$. It is easy to see that the Sylow 11-subgroup of H is normal in H by Lemma 3.

By Lemma 1, there is a character $\chi \in Irr(H)$ such that the degree of χ divides $|H : O_{11}(H)|$, a contradiction. Hence $N_G(M) > C_G(M)$. It follows that $N_G(M)/C_G(M)$ is isomorphic to either a 2-group or a 5-group or a {2, 5}-group.

If $N_G(M)/C_G(M)$ is a 5-group, then $G/C_G(M) \cong Z_5$. It follows that $|C_G(M)| = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$. But by Lemma 3, the Sylow 11-subgroup G_{11} of $C_G(M)$ is normal in $C_G(M)$. Since $C_G(M)$ is characteristic in G, then G_{11} is normal in G, a contradiction.

If $N_G(M)/C_G(M)$ is a 2-group, then similarly as above arguments, we also can get that G_{11} is normal in G. So we rule out this case.

Similarly we can rule out the case when $N_G(M)/C_G(M)$ is a {2, 5}-group.

Let p=2. Then $|M| = 2^a$ where $a \in \{1, 2, 3, 4, 5, 6\}$. Similarly as p=5, we by [8], $G/C_G(M) = N_G(M)/C_G(M)$ is isomorphic to a subgroup of GL(a, 2). It follows that the primes 5 and 11 divide the order of $C_G(M)$. Similarly, we can rule out this case since the Sylow 11-subgroup of $C_G(M)$ is normal in $C_G(M)$ and $C_G(M)$ is characteristic in G.

Therefore G is insoluble. So by Lemma 2, G has a normal series $1 \lhd H \lhd K \lhd G$ such that K/H is direct product of isomorphic non-abelian simple group and $|G/K| \,|\, |Out(K/H)|$.

By [9, 10], we have that the order of Aut(K/H) is divisible by neither 7 nor 11. If 7 or 11 divides the order of H. Then by Lemma 1, there is a character χ with that $7 \,|\, \chi(1) \,|\, |K : G_7|$ or $11 \,|\, \chi(1) \,|\, |K : G_{11}|$, respectively. Also get a contradiction. Hence the primes 7 and 11 divide the order of K/H. By Lemma 4, we have that K/H is isomorphic to M_{22} or A_{11}.

Let K/H is isomorphic to M_{22}. Then $M_{22} \le G/H \le Aut(M_{22})$. If G/H is isomorphic to $Aut(M_{22})$. Then order consideration rules out. Hence G/H is isomorphic to M_{22} and $|H| = 3^2 \cdot 5$. By [7], $cd(M_{22}) = \{1, 21, 45, 99, 154, 210, 231, 280, 385\}$ and so the graph $\Gamma(M_{22})$ of M_{22} is complete. It follows that G is isomorphic to a product HM_{22} of H and M_{22}.

Let K/H is isomorphic to A_{11}. Then $A_{11} \le G/H \le Aut(A_{11})$. Since $Out(A_{11}) = 2$, then G/H is isomorphic to A_{11}. Order consideration means that G is isomorphic to A_{11}.

This completes the proof of the main theorem.

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